6.851 | Lecture 18 | Apr. 26, 2012

TODAY: Succinct data structures II (of 2)

- compact & succinct suffix arrays/trees
- $O(T \lg\lg T)$-bit suffix array
- $O(T)$-bit suffix array
- succinct suffix array → tree

Compact suffix arrays/trees:

TODAY! * Grossi & Vitter: [STOC 2000; SICOMP 2005] first

$$(1+1/\varepsilon)\underbrace{|T|\lg|\Sigma|}_{\text{OPT*}} + \underbrace{2|T|}_{\text{if } |\Sigma|>2} + O\left(\frac{T}{\lg\lg T}\right) \text{ bits}$$

*without compression

$$O\left(\frac{P}{\log_\Sigma T} + |\text{output}|\cdot\log_\Sigma^\varepsilon T\right) \text{ query}$$

- Ferragina & Manzini: [FOCS 2000; JACM 2005]

$$5\cdot\underbrace{H_k(T)}_{\text{kth order empirical entropy}}\cdot|T| + O\left(T\frac{\Sigma+\lg T}{\lg\lg T} + T^\varepsilon\Sigma^{\Sigma+1}\right)$$

Burrows–Wheeler Trans.

for any fixed k

$$= \sum_{|w|=k} \underbrace{\Pr\{w \text{ occurring}\}}_{=\#\text{occurrences}/|T|}\cdot H_0(\underbrace{\text{string of successor characters of } w}_{=\sum_x p_x \lg\frac{1}{p_x},\ p_x=\frac{\#\text{occ.}(wx)}{\#\text{occ.}(w)}})$$

= # bits/char. in OPT code depending on last k chars

$$O(|P| + |\text{output}|\cdot\lg^\varepsilon T) \text{ query}$$

- Sadakane: [J. Alg. 2003] large alphabets

$$\frac{1+\varepsilon'}{\varepsilon}\cdot H_0(T)\cdot|T| + O(T\lg\lg\Sigma + \Sigma\lg\Sigma) \text{ bits}$$

$$O(P\lg T + |\text{output}|\cdot\lg^\varepsilon T) \text{ query for } 0<\varepsilon,\varepsilon'<1$$

Succinct:

- Grossi, Gupta, Vitter: [SODA 2003]
  $H_k(T)\cdot|T| + O\left(T \lg \Sigma_i \cdot \frac{\lg\lg T}{\lg T}\right)$ bits
  $O\left(P \lg \Sigma_i + \frac{\lg^2 T}{\lg\lg T}\lg\Sigma_i\right)$ query
- Ferragina, Manzini, Mäkinen, Navarro: [TALG 2007]
  $H_k(T)\cdot|T| + O(T/\lg^{\varepsilon} n)$ bits
  $O(P + |\text{output}|\cdot\lg^{1+\varepsilon} T)$ query + $O(P)$ counting query

Extras:

- $\underbrace{\text{low-space}}_{O(|T|\lg|\Sigma|)\text{ working}}$ construction [Hon, Sadakane, Sung - FOCS 2003; Hon, Lam, Sadakane, Sung, Yu - Alg. 2007]
- suffix-tree ops. like maximal common substrings [Hon & Sadakane - CPM 2002; Sadakane - TCS 2007]
- document retrieval [Sadakane - J. Disc. Alg. 2007]
- dynamic [Chan, Hon, Lam, Sadakane - T. Alg. 2007]
- implementations & experiments [...]

Compressed suffix arrays: [Grossi & Vitter – STOC 2000]

follow divide & conquer of linear-time suffix tree/array alg. [L16], but 2-way instead of 3-way

- **start:** text $T_0 = T$
  size $n_0 = n$
  suffix array $SA_0 = SA$
  $SA[i]$ = index in T of $\overbrace{\text{ith}}^{\text{in sorted order of suffixes}}$ suffix of T
- **step:** $T_{k+1} = \langle (T_k[2i], T_k[2i+1]) \text{ for } i = 0, 1, \ldots, n/2 \rangle$
  $n_{k+1} = n_k/2 = n/2^k$
  $SA_{k+1} = \frac{1}{2} \cdot$ extract even entries of SA_k
- represent SA_k using SA_{k+1}:
  ① **even-succ**$_k(i) = \begin{cases} i & \text{if } SA_k[i] \text{ is even} \\ j & \text{if } SA_k[i] = SA_k[j] - 1 \text{ is odd} \end{cases}$
  (jth suffix starts right after ith suffix)
  ② **even-rank**$_k(i)$ = # even suffixes preceding ith suffix
  = # even values in $SA_k[:i]$
  ③ $$SA_k[i] = \underbrace{2 \cdot \underbrace{SA_{k+1}[\underbrace{\text{even-rank}_k(\underbrace{\text{even-succ}_k(i)}_{\text{round to even suffix} \Rightarrow \text{ in } SA_{k+1}})}_{\text{name of even suffix in } SA_{k+1}}]}_{\text{index of even suffix in } T_{k+1}}}_{\text{index of even suffix in } T_k} - \underbrace{(1 - \text{is-even}_k(i))}_{\text{unround}}$$
- stop recursion at level $\ell = \lg \lg n \Rightarrow n_\ell = n/\lg n$
  $\Rightarrow$ can afford naïve ptr. encoding: $n_\ell \lg n_\ell \le n$ bits

$\Rightarrow O(\lg \lg n)$ query to $SA[i]$, given $O(1)$-time is-even-suffix & even-succ & even-rank

$\underline{\text{is-even-suffix}_k}(i) = \begin{cases} 1 & \text{if } SA_k[i] \text{ is even} \\ 0 & \text{else} \end{cases} \rightarrow n_k \text{ bits}$

$\underline{\text{even-rank}_k} = \text{rank}_1 \text{ in is-even-suffix} \rightarrow O\left(n_k \frac{\lg\lg n_k}{\lg n_k}\right) \text{ bits}$ [L17]

$\underline{\text{even-succ}_k}$:

- trivial for $SA_k[i]$ even ($n_k/2$ such i's)
- store answers for $SA_k[i]$ odd ($n_k/2$ such i's)
- order by $i \Rightarrow \text{even-succ}_k(i) = \text{odd-answers}[\underbrace{\underline{\text{odd-rank}_k(i)}}_{= i - \text{even-rank}_k(i)}]$

$= \text{order by odd suffix } T_k[SA_k[i]:]$

$= \text{order by } (T_k[SA_k[i]], T_k[\underbrace{SA_k[i]+1}_{\text{even}}:])$

(def. of even-succ)

$= \text{order by } (T_k[SA_k[i]], T_k[SA_k[\text{even-succ}_k(i)]:])$

$= \text{order by } (T_k[SA_k[i]], \underbrace{\text{even-succ}_k(i)}_{\text{answer!}})$ (SA_k is a suffix array)

- actually store these pairs, in order by value
- assume $|\Sigma| = 2$ here

$\Rightarrow$ storing sorted array of $n_k/2$ values, $2^k + \lg n_k$ bits each

- store leading $\lg n_k$ bits of each value v_i via unary differential encoding: $0^{\text{lead}(v_1)} 1 0^{\text{lead}(v_2) - \text{lead}(v_1)} 1 \ldots$
  - $\Rightarrow n_k/2$ 1's & $\leq 2^{\lg n_k} = n_k$ 0's $\Rightarrow \leq \frac{3}{2} n_k$ bits ↗[L17]
  - $\text{lead}(v_i) = \underline{\text{rank}_0}(\underline{\text{select}_1}(i)) - \underline{\text{rank}_0}(\underline{\text{select}_1}(i-1))$
- store trailing 2^k bits of each v_i explicitly in array
  - $\Rightarrow 2^k \cdot n_k/2 = n/2$ bits

$\Rightarrow \frac{1}{2} n + \underbrace{\frac{3}{2} n_k + O\left(\frac{n_k}{\lg\lg n_k}\right)}_{\text{rank \& select}} \text{ bits}$

$$\underline{\text{Total}}: \sum_{k=0}^{\lg\lg n} \left(n_k + \frac{1}{2} n + \frac{3}{2} n_k + O\left(\frac{n_k}{\lg\lg n_k}\right)\right)$$

$$= \frac{1}{2} n \underline{\lg\lg n} + 5n + O\left(\frac{n}{\lg\lg n}\right) \text{ bits}$$ — PROGRESS ~ BUT TOO BIG

Compact suffix array: $O(n)$ bits

- store only $1/\varepsilon + 1$ levels of recursion:
  $0, \varepsilon\ell, 2\varepsilon\ell, \dots, \ell = \lg\lg n$
- represent $SA_{k\varepsilon\ell}$ using $SA_{(k+1)\varepsilon\ell}$ similarly:
  - "even" → "divisible by $2^{\varepsilon\ell}$" = "in $SA_{(k+1)\varepsilon\ell}$"
  - is-even$_k$ = $n_{k\varepsilon\ell}$-bit vector as before + rank$_1$ struct.
  - $\text{succ}_k(i) = j$ where $SA_{k\varepsilon\ell}[i] = SA_{k\varepsilon\ell}[j] - 1$
    stored in $n + 2n_{k\varepsilon\ell} + O\left(\frac{n_{k\varepsilon\ell}}{\lg\lg n_{k\varepsilon\ell}}\right)$ bits like even-succ
    ~ except can't skip ½ the values
- to compute $SA_{k\varepsilon\ell}[i]$:
  ① follow succ$_k$ repeatedly until j $\underline{\text{in } SA_{(k+1)\varepsilon\ell}}$ → is-even$_k$ bit vector
  ② recurse: $SA_{(k+1)\varepsilon\ell}[\underline{\text{rank}_1}(j)]$ → is-even$_k$
  ③ multiply by $2^{\varepsilon\ell}$, subtract by # steps in ①
- $\le 2^{\varepsilon\ell} = \lg^{\varepsilon} n$ succ$_k$ calls per level

$\Rightarrow O(\lg^{\varepsilon} n \lg\lg n) = O(\lg^{\varepsilon'} n)$ query to $SA[i]$

- space: $\displaystyle\sum_{k=0}^{1/\varepsilon} \left(\underbrace{n_{k\varepsilon\ell}}_{\text{is-even}_k} + \underbrace{n + 2n_{k\varepsilon\ell}}_{\text{succ}_k} + O\left(\frac{n_{k\varepsilon\ell}}{\lg\lg n_{k\varepsilon\ell}}\right)\right)$

  $= \left(\underset{+1}{1/\varepsilon} + 6\right)n + O\left(\frac{n}{\lg\lg n}\right)$ bits
- optimizations:
  - succ$_k$ free at level $k=0$ $\Rightarrow \Sigma \le 4n_{\varepsilon\ell} = O\left(\frac{n}{\lg^{\varepsilon} n}\right)$
  - store is-even$_k$ as $\underline{\text{succinct dictionary (with rank)}}$ → [Brodnik & Munro] of L17
    $\Rightarrow \lg\binom{n_{k\varepsilon\ell}}{n_{(k+1)\varepsilon\ell}} + o \sim n_{(k+1)\varepsilon\ell} \lg \underbrace{\frac{n_{k\varepsilon\ell}}{n_{(k+1)\varepsilon\ell}}}_{2^{\varepsilon\ell}} + o = O\left(n\frac{\lg\lg n}{\lg^{\varepsilon} n}\right)$

$\Rightarrow (1/\varepsilon + 1)\cdot n + O\left(\frac{n}{\lg\lg n}\right)$ bits

$\boxed{\text{OPEN}}$: $O(n)$ bits & $o(\lg^{\varepsilon} n)$ query

Compact suffix tree given suffix array:

[Munro, Raman, Rao – J.Alg. 2001]

- store compressed binary trie on $2n+1$ nodes in $4n+o(n)$ bits using balanced parens. [L17]

↪ "skipping the skips": can't store edge lengths

- during search for P, maintain letter depth of node:
  to descend $\textcircled{x} \to \textcircled{y}$:
  - compute length of edge by finding longest match between leftmost-leaf(y) & rightmost-leaf(y), starting at letter depth of x (leaf rank + SA)
  - check that P matches these letters

$\Rightarrow O((P + |\text{output}|) \cdot \text{cost for SA lookup})$

↪ # matches via # leaves in subtree

↪ enumerate k matches via leaf select + rank

Grossi & Vitter achieve $O\left(\frac{P}{\log_\Sigma T} + |\text{output}| \cdot \log_\Sigma^\varepsilon T\right)$ ↖ word RAM

Leaf ops:

leaf = ())

leaf-rank = $\text{rank}_{())}(\text{here})$

leaf-select = $\text{select}_{())}(i)$

leaves in subtree = $\text{rank}_{())}(\text{matching) of parent}) - \text{rank}_{())}(\text{here})$

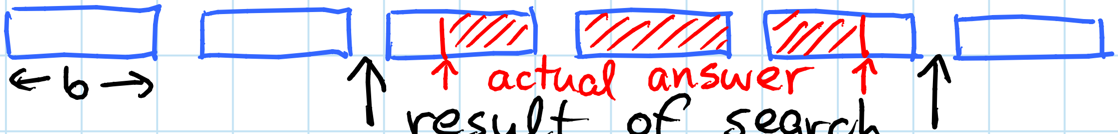
leftmost-leaf = leaf-select(leaf-rank(here) + 1)

rightmost-leaf = leaf-select(leaf-rank(matching) of parent) − 1))

"Succinct" suffix tree, given suffix array

[Munro, Raman, Rao; fixed]

- use suffix tree above on every bth suffix in suffix order (keep every bth leaf in tree)
- search in tree narrows to interval of size-b blocks in SA:



⇒ need to find first & last match in a block

- lookup table:
  - for any b b-bit strings in sorted order 2^{b^2}
  - & any ≤ b-bit query string, 2^b
  - find first & last prefix match $O(\lg b)$

  ⇒ $O(2^{b^2+b} \lg b)$ bits $= O(\sqrt{n})$ if $b \leq \frac{1}{2}\sqrt{\lg n}$
- to search in a block:
  - ⓪ locate the b suffixes in T using SA $O(b) +$
  - ① read next b bits from P & from all b suffixes (in T) $\big(O(1) + O(b)\big)$
  - ② repeat $\lceil |P|/b \rceil$ times $\cdot\, O(P/b)$

$$\Rightarrow O(P + \underbrace{b}_{\text{forgotten in [MRR]}}) \text{ time } = O(P + \lg\lg n) \text{ if } b = O(\lg\lg n)$$

× cost of SA query ⇒ $O(P \lg^{\varepsilon} n)$

- $O(\frac{n}{b})$ bits plus size of suffix array

6.851 Advanced Data Structures
Spring 2012