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5.8 The Harmonic Oscillator

To illustrate the beauty and efficiency in describing the dynamics of a quantum system using the dirac notation and operator algebra, we reconsider the one-dimensional harmonic oscillator discussed in section 4.4.2 and described by the Hamiltonian operator

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2} K \; \mathbf{x}^2, \tag{5.127}$$

with

$$[\mathbf{x}, \mathbf{p}] \; = \mathrm{j}\hbar. \tag{5.128}$$

5.8.1 Energy Eigenstates, Creation and Annihilation Operators

It is advantageous to introduce the following normalized position and momentum operators

$$\mathbf{X} \;\; = \;\; \sqrt{\frac{K}{\hbar\omega_0}} \, \mathbf{x} \tag{5.129}$$

$$\mathbf{P} \;\; = \;\; \sqrt{m\hbar\omega_0} \, \mathbf{p} \tag{5.130}$$

with $\omega_0 = \sqrt{\frac{K}{m}}$. The Hamiltonian operator and the commutation relationship of the normalized position and momentum operator resume the simpler forms

$$\mathbf{H} = \frac{\hbar\omega_0}{2}\left(\mathbf{P}^2 + \mathbf{X}^2\right), \tag{5.131}$$

$$[\mathbf{X}, \mathbf{P}] = \mathrm{j}\,. \tag{5.132}$$

Algebraically, it is very useful to introduce the nonhermitian operators

$$\mathbf{a} = \frac{1}{\sqrt{2}}\left(\mathbf{X} + \mathrm{j}\mathbf{P}\right), \tag{5.133}$$

$$\mathbf{a}^+ = \frac{1}{\sqrt{2}}\left(\mathbf{X} - \mathrm{j}\mathbf{P}\right), \tag{5.134}$$

which satisfy the commutation relation

$$\left[\mathbf{a}, \mathbf{a}^+\right] = 1. \tag{5.135}$$

We find

$$\mathbf{a}\mathbf{a}^+ = \frac{1}{2}\left(\mathbf{X}^2 + \mathbf{P}^2\right) - \frac{\mathrm{j}}{2}[\mathbf{X}, \mathbf{P}] = \frac{1}{2}\left(\mathbf{X}^2 + \mathbf{P}^2 + 1\right), \tag{5.136}$$

$$\mathbf{a}^+\mathbf{a} = \frac{1}{2}\left(\mathbf{X}^2 + \mathbf{P}^2\right) + \frac{\mathrm{j}}{2}[\mathbf{X}, \mathbf{P}] = \frac{1}{2}\left(\mathbf{X}^2 + \mathbf{P}^2 - 1\right), \tag{5.137}$$

and the Hamiltonian operator can be rewritten in terms of the new operators $\mathbf{a}$ and $\mathbf{a}^+$ as

$$\mathbf{H} = \frac{\hbar\omega_0}{2}\left(\mathbf{a}^+\mathbf{a} + \mathbf{a}\mathbf{a}^+\right) \tag{5.138}$$

$$= \hbar\omega_0\left(\mathbf{a}^+\mathbf{a} + \frac{1}{2}\right). \tag{5.139}$$

We introduce the operator

$$\mathbf{N} = \mathbf{a}^+\mathbf{a}, \tag{5.140}$$

which is a hermitian operator. Up to an additive constant 1/2 and a scaling factor $\hbar\omega_0$ equal to the energy of one quantum of the harmonic oscillator it is equal to the Hamiltonian operator of the harmonic oscillator. Obviously, $\mathbf{N}$ is the number operator counting the number of energy quanta excited in a harmonic oscillator. We assume that the number operator $\mathbf{N}$ has eigenvectors denoted by $|n\rangle$ and corresponding eigenvalues N_n

$$\mathbf{N}\ |n\rangle = \mathbf{a}^{+}\mathbf{a}\ |n\rangle = N_n\,|n\rangle\,. \tag{5.141}$$

We also assume that these eigenvectors are normalized and since $\mathbf{N}$ is hermitian they are also orthogonal to each other

$$\langle m\,|n\rangle = \delta_{mn}. \tag{5.142}$$

Multiplication of this equation with the operator $\mathbf{a}$ and use of the commutation relation (5.135) leads to

$$\mathbf{a}\,\mathbf{a}^{+}\mathbf{a}\,|n\rangle = N_n\mathbf{a}\,|n\rangle \tag{5.143}$$
$$\left(\mathbf{a}^{+}\mathbf{a}+\mathbf{1}\right)\mathbf{a}\,|n\rangle = N_n\mathbf{a}\,|n\rangle \tag{5.144}$$
$$\mathbf{N}\ \mathbf{a}\,|n\rangle = (N_n-1)\,\mathbf{a}\,|n\rangle \tag{5.145}$$

Eq.(5.143) indicates that if $|n\rangle$ is an eigenstate to the number operator $\mathbf{N}$ then the state $\mathbf{a}\ |n\rangle$ is a new eigenstate to $\mathbf{N}$ with eigenvalue N_n-1. Because of this property, the operator $\mathbf{a}$ is called a lowering operator or annihilation operator, since application of the annihilation operator to an eigenstate with N_n quanta leads to a new eigenstate that contains one less quantum

$$\mathbf{a}\,|n\rangle = C\,|n-1\rangle\,, \tag{5.146}$$

where C is a yet undetermined constant. This constant follows from the normalization of this state and being an eigenvector to the number operator.

$$\langle n|\,\mathbf{a}^{+}\mathbf{a}\,|n\rangle = |C|^2\,, \tag{5.147}$$
$$C = \sqrt{n}. \tag{5.148}$$

Thus

$$\mathbf{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \tag{5.149}$$

Clearly, if there is a state with $n=0$ application of the annihilation operator leads to the null-vector in this Hilbert space, i.e.

$$\mathbf{a}\,|0\rangle = 0, \tag{5.150}$$

and there is no other state with a lower number of quanta, i.e. $N_0=0$ and $N_n=n$. This is the ground state of the harmonic oscillator, the state with the lowest energy.

If $\mathbf{a}$ is an annihilation operator for energy quanta, $\mathbf{a}^+$ must be a creation operator for energy quanta, otherwise the state $|n\rangle$ would not fulfill the eigenvalue equation Eq.(5.141)

$$\begin{aligned} \mathbf{a}^+\mathbf{a}\,|n\rangle &= n\,|n\rangle && (5.151)\\ \mathbf{a}^+\sqrt{n}\,|n-1\rangle &= n\,|n\rangle && (5.152)\\ \mathbf{a}^+\,|n-1\rangle &= \sqrt{n}\,|n\rangle && (5.153) \end{aligned}$$

or

$$\mathbf{a}^+\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \ . \qquad (5.154)$$

Starting from the energy ground state of the harmonic oscillator $|0\rangle$ with energy $\hbar\omega_0/2$ we can generate the n-th energy eigenstate by n-fold application of the creation operator $\mathbf{a}^+$ and proper normalization

$$|n\rangle = \frac{1}{\sqrt{(n+1)!}}\ \left(\mathbf{a}^+\right)^n|0\rangle\,, \qquad (5.155)$$

with

$$\mathbf{H}\,|n\rangle = E_n\,|n\rangle\,, \qquad (5.156)$$

and

$$E_n\ = \hbar\omega_0\left(n+\frac{1}{2}\right). \qquad (5.157)$$

5.8.2 Matrix Representation

We can express the normalized position and momentum operators as functions of the creation and annihilation operators

$$\begin{aligned} \mathbf{X} &= \frac{1}{\sqrt{2}}\left(\mathbf{a}^+ + \mathbf{a}\right)\,, && (5.158)\\ \mathbf{P} &= \frac{\mathrm{j}}{\sqrt{2}}\left(\mathbf{a}^+ - \mathbf{a}\right)\,. && (5.159) \end{aligned}$$

These operators do have the following matrix representations

$$\begin{aligned} \langle m|\,\mathbf{a}\,|n\rangle &= \sqrt{n}\delta_{m,n-1}\,, \qquad \langle m|\ \mathbf{a}^+\ |n\rangle = \sqrt{n+1}\delta_{m,n+1}\,, && (5.160)\\ \langle m|\,\mathbf{a}^+\mathbf{a}\,|n\rangle &= n\delta_{m,n}\,, \qquad \langle m|\ \mathbf{a}\mathbf{a}^+\ |n\rangle = (n+1)\,\delta_{m,n}\,, && (5.161)\\ \langle m|\,\mathbf{X}\,|n\rangle &= \frac{1}{\sqrt{2}}\left(\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}\right)\,, && (5.162)\\ \langle m|\,\mathbf{P}\,|n\rangle &= \frac{\mathrm{j}}{\sqrt{2}}\left(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}\right)\,, && (5.163) \end{aligned}$$

$$\langle m|\,\mathbf{a}^2\,|n\rangle = \sqrt{n(n-1)}\delta_{m,n-2}\,, \tag{5.164}$$

$$\langle m|\,\mathbf{a}^{+2}\,|n\rangle = \sqrt{(n+1)\,(n+2)}\delta_{m,n+2}\,, \tag{5.165}$$

$$\langle m|\,\mathbf{X}^2\,|n\rangle = \frac{1}{2}\left(\begin{array}{c} (2n+1)\delta_{m,n} + \sqrt{n(n-1)}\delta_{m,n-2} \\ +\sqrt{(n+1)\,(n+2)}\delta_{m,n+2} \end{array}\right), \tag{5.166}$$

$$\langle m|\,\mathbf{P}^2\,|n\rangle = \frac{1}{2}\left(\begin{array}{c} (2n+1)\delta_{m,n} - \sqrt{n(n-1)}\delta_{m,n-2} \\ -\sqrt{(n+1)\,(n+2)}\delta_{m,n+2} \end{array}\right). \tag{5.167}$$