7.3.6 Homogeneous and Inhomogeneous Broadening

Laser media are also distinguished by the line broadening mechanisms involved. Very often it is the case that the linewidth observed in the absorption or emission spectrum is not only due to dephasing process that are acting on

Laser Medium	Wave-length $\lambda_0(n\text{m})$	Cross Section σ (cm^2)	Upper-St. Lifetime τ_L (μs)	Linewidth $\Delta f_{FWHM} = \frac{2}{T_2}$(THz)	Typ	Refr. index n
Nd^{3+}:YAG	1,064	$4.1 \cdot 10^{-19}$	1,200	0.210	H	1.82
Nd^{3+}:LSB	1,062	$1.3 \cdot 10^{-19}$	87	1.2	H	1.47 (ne)
Nd^{3+}:YLF	1,047	$1.8 \cdot 10^{-19}$	450	0.390	H	1.82 (ne)
Nd^{3+}:YVO_4	1,064	$2.5 \cdot 10^{-19}$	50	0.300	H	2.19 (ne)
Nd^{3+}:glass	1,054	$4 \cdot 10^{-20}$	350	3	H/I	1.5
Er^{3+}:glass	1,55	$6 \cdot 10^{-21}$	10,000	4	H/I	1.46
Ruby	694.3	$2 \cdot 10^{-20}$	1,000	0.06	H	1.76
Ti^{3+}:Al_2O_3	660-1180	$3 \cdot 10^{-19}$	3	100	H	1.76
Cr^{3+}:LiSAF	760-960	$4.8 \cdot 10^{-20}$	67	80	H	1.4
Cr^{3+}:LiCAF	710-840	$1.3 \cdot 10^{-20}$	170	65	H	1.4
Cr^{3+}:LiSGAF	740-930	$3.3 \cdot 10^{-20}$	88	80	H	1.4
He-Ne	632.8	$1 \cdot 10^{-13}$	0.7	0.0015	I	$\sim$1
Ar^+	515	$3 \cdot 10^{-12}$	0.07	0.0035	I	$\sim$1
CO_2	10,600	$3 \cdot 10^{-18}$	2,900,000	0.000060	H	$\sim$1
Rhodamin-6G	560-640	$3 \cdot 10^{-16}$	0.0033	5	H	1.33
semiconductors	450-30,000	$\sim 10^{-14}$	$\sim$ 0.002	25	H/I	3 - 4

Table 7.1: Wavelength range, cross-section for stimulated emission, upper-state lifetime, linewidth, typ of lineshape (H=homogeneously broadened, I=inhomogeneously broadened) and index for some often used solid-state laser materials, and in comparison with semiconductor and dye lasers.

all atoms in the same, i.e. homogenous way. Lattice vibrations that lead to a line broadening of electronic transisitions of laser ions in the crystal act in the same way on all atoms in the crystal. Such mechanisms are called homogeneous broadening. However, It can be that in an atomic ensemble there are groups of atoms with a different center frequency of the atomic transistion. The overall ensemble therefore may eventually show a very broad linewidth but it is not related to actual dephasing mechanism that acts upon each atom in the ensemble. This is partially the case in Nd:silicate glass lasers, see table 7.1 and the linewidth is said to be inhomogeneously broadened. Wether a transistion is homogenously or inhomogeneously broadened can be tested by using a laser to saturate the medium. In a homogenously

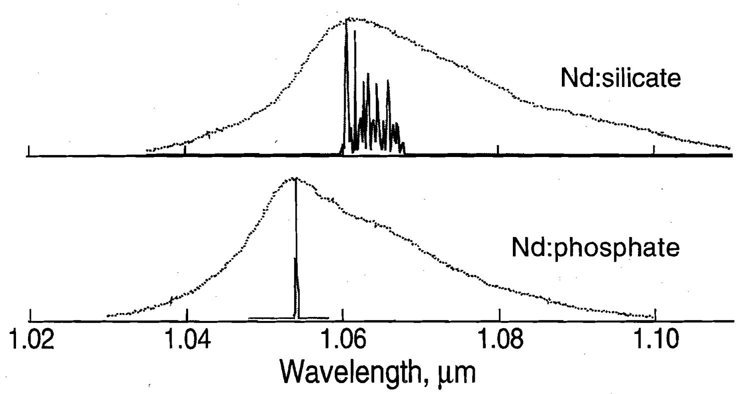

Figure 7.18: Laser with inhomogenously broaden laser medium (Nd:silicate glass) and homogenously broadened laser medium (Nd:phosphate glass), [14]

broadened medium the loss or gain saturates homogenously, i.e. the whole line is reduced. In an inhomogenously broadened medium a spectral hole burning occurs, i.e. only that sub-group of atoms that are sufficiently in resonance with the driving field saturate and the others not, which leads to a hole in the spectral distribution of the atoms. Figure 7.18 shows the impact of an inhomogeneously broadened gain medium on the continous wave output spectrum of a laser. In homogenous broadening leads to lasing of many longitudinal laser modes because inhomogenous saturation of the gain. In the homogenously broadened medium the gain saturates homogenously and only one or a few modes can lase. An important inhomogenous broadening mechanism in gases is doppler broadening. Due to the motion of the atoms in a gas relative to an incident electromagnetic beam, the center frequency of each atomic transistion is doppler shifted according to its velocity by

$$f = \left(1 \pm \frac{v}{c}\right) f_0, \tag{7.5}$$

where the plus sign is correct for an atom moving towards the beam and the minus sign for a atom moving with the beam. The velocity distribution of an ideal gas with atoms or molecules of mass m in thermal equilibrium is given by the Maxwell-Boltzman distribution

$$p(v) = \sqrt{\frac{m}{2\pi kT}} \exp\left(-\frac{mv^2}{2kT}\right). \tag{7.6}$$

This means that $p(v)dv$ is equal to the brobability that the atom or molecule has a velocity in the interval $[v, v+dv]$. Here, v is the component of the velocity that is in the direction of the beam. If the homogenous linewidth of the atoms is small compared to the doppler broading, we obtain the lineshape of the inhomogenously broadened gas simply by substituting the velocity by the induced frequency shift due to the motion

$$v = c\,\frac{f - f_0}{f_0} \tag{7.7}$$

Then the lineshape is a Gaussian

$$g(f) = \sqrt{\frac{mc}{2\pi kT f_0}}\exp\left[-\frac{mc^2}{2kT}\left(\frac{f-f_0}{f_0}\right)^2\right]. \tag{7.8}$$

The full width at half maximum of the line is

$$\Delta f = 8\ln(2)\sqrt{\frac{kT}{mc^2}}f_0. \tag{7.9}$$

7.4 Laser Dynamics (Single Mode)

In this section we want study the single mode laser dynamics. The laser typically starts to lase in a few closely spaced longitudinal modes, which are incoherent with each other and the dynamics is to a large extent similar to the dynamics of a single mode that carries the power of all lasing modes. To do so, we complement the rate equations for the populations in the atomic medium, that can be reduced to the population of the upper laser level Eq.(7.4) as discussed before with a rate equation for the photon population in the laser mode.

There are two different kinds of laser cavities, linear and ring cavities, see Figure 7.19

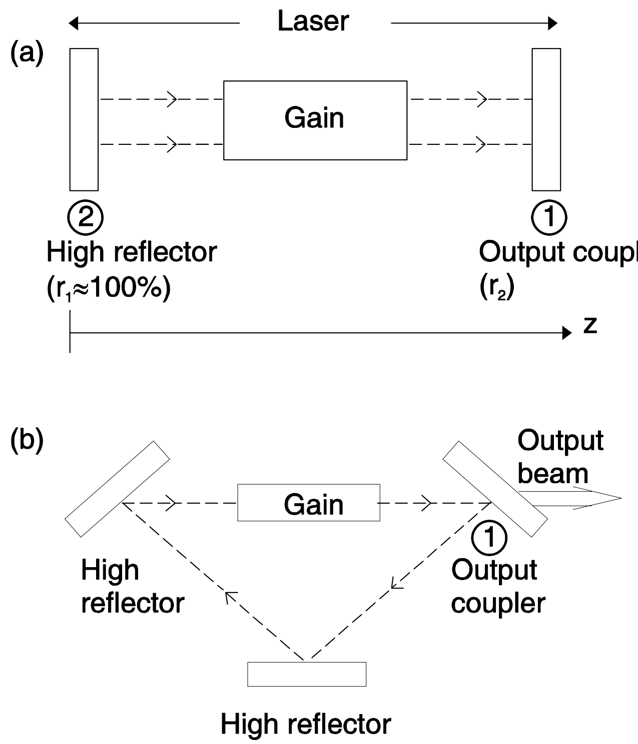

Figure 7.19: Possible cavity configurations. (a) Schematic of a linear cavity laser. (b) Schematic of a ring laser.

The laser resonators can be modelled as Fabry Perots as discussed in section . Typically the techniques are used to avoid lasing of transverse modes and only the longitudinal modes are of interest. The resonance frequencies of the longitudinal modes are determined by the round trip phase to be a multiple of 2π

$$\phi(\omega_m) = 2m\pi. \tag{7.10}$$

neighboring modes are space in frequency by the inverse roundtrip time

$$\phi(\omega_0 + \Delta\omega) = \phi(\omega_0) + T_R\Delta\omega = 2m\pi. \tag{7.11}$$

T_R is the round trip time in the resonator, which is

$$T_R = \frac{2^* L}{\nu_g}, \tag{7.12}$$

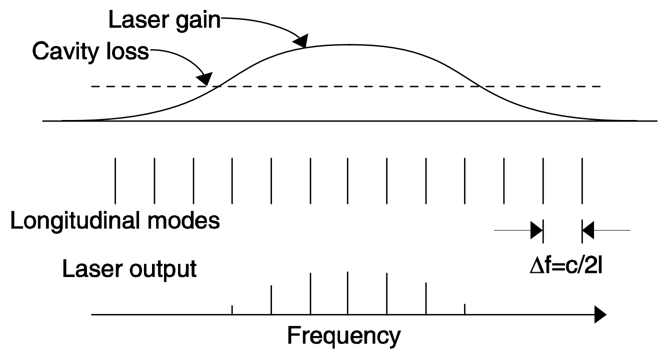

Figure 7.20: Laser gain and cavity loss spectra, longitudinal mode location, and laser output for multimode laser operation.

where ν_g is the group velocity in the cavity in the frequency range considered, and L is the cavity length of the linear or ring cavity and $2^* = 1$ for the ring cavity and $2^* = 2$ for the linear cavity. In the case of no dispersion, the longitudinal modes of the resonator are multiples of the inverse roundtrip timeand

$$f_m = m \frac{1}{T_R}. \tag{7.13}$$

The mode spacing of the longitudinal modes is

$$\Delta f = f_m - f_{m-1} = \frac{1}{T_R} \tag{7.14}$$

If we assume frequency independent cavity loss and Lorentzian shaped gain (see Fig. 7.20). Initially when the laser gain is larger then the cavity loss, many modes will start to lase. To assure single frequency operation a filter (etalon) can be inserted into the laser resonator, see Figure 7.21. If the laser is homogenously broadened the laser gain will satured to the loss level and only the mode at the maximum of the gain will lase. If the gain is not homogenously broadend and in the absence of a filter many modes will lase.

For the following we assume a homogenously broadend laser medium and only one cavity mode is able to lase. We want to derive the equations of motion for th population inversion, or population in the upper laser level and the photon number in that mode, see Figure 7.22.

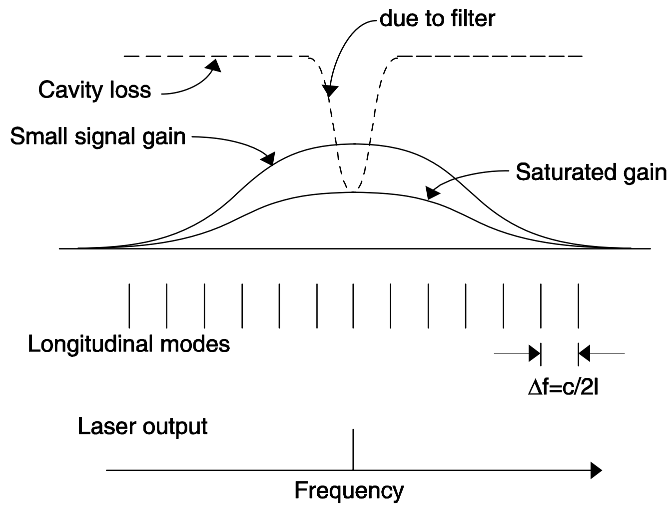

Figure 7.21: Gain and loss spectra, longitudinal mode locations, and laser output for single mode laser operation.

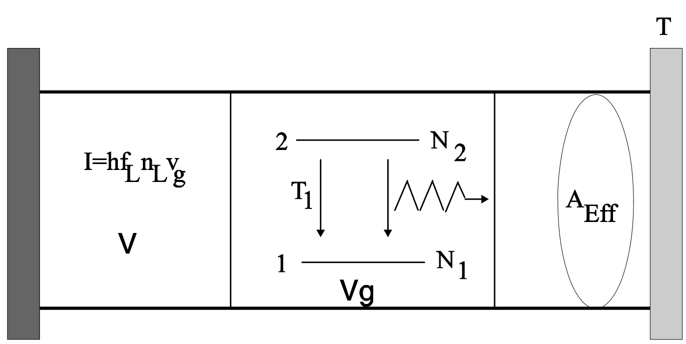

Figure 7.22: Rate equations for a laser with two-level atoms and a resonator.

The intensity I in a mode propagating at group velocity v_g with a mode volume V is related to the number of photons N_L or the number density $n_L = N_L/V$ stored in the mode with volume V by

$$I = hf_L \frac{N_L}{2^* V} v_g = \frac{1}{2^*} hf_L n_L v_g, \tag{7.15}$$

where hf_L is the photon energy. $2^* = 2$ for a linear laser resonator (then only half of the photons are going in one direction), and $2^* = 1$ for a ring laser. In this first treatment we consider the case of space-independent rate equations, i.e. we assume that the laser is oscillating on a single mode and pumping and mode energy densities are uniform within the laser material. With the interaction cross section σ for stimulated emission defined as

$$\sigma = \frac{hf_L}{I_s T_1}, \tag{7.16}$$

and Eq. (7.4) with the number of atoms in the mode, we obtain

$$\frac{d}{dt}N_2 = -\frac{N_2}{\tau_L} - 2^*\sigma N_2 v_g n_L + R_p. \tag{7.17}$$

Here, $v_g n_L$ is the photon flux, σ is the stimulated emission cross section, $\tau_L = \gamma_{21}$ the upper state lifetime and R_p is the pumping rate into the upper laser level. A similar rate equation can be derived for the photon density

$$\frac{d}{dt}n_L = -\frac{n_L}{\tau_p} + 2^*\frac{\sigma v_g}{V}N_2\left(n_L + \frac{1}{V}\right). \tag{7.18}$$

Here, τ_p is the photon lifetime in the cavity or cavity decay time. The $1/V$ -term in Eq.(7.18) accounts for spontaneous emission which is equivalent to stimulated emission by one photon occupying the mode with mode volume V. For a laser cavity with a semi-transparent mirror with amplitude transmission T, see section 2.3.8, producing a power loss $2l = 2T$ per round-trip in the cavity, the cavity decay time is $\tau_p = 2l/T_R$, if $T_R = 2^*L/c_0$ is the roundtrip-time in linear cavity with optical length $2L$ or a ring cavity with optical length L. Eventual internal losses can be treated in a similar way and contribute to the cavity decay time. Note, the decay rate for the inversion in the absence of a field, $1/\tau_L$, is not only due to spontaneous emission, but is also a result of non radiative decay processes. See for example the four level system shown in Fig. 7.6.

So the two rate equations are

$$\frac{d}{dt}N_2 = -\frac{N_2}{\tau_L} - 2^*\sigma v_g N_2 n_L + R_p \tag{7.19}$$

$$\frac{d}{dt}n_L = -\frac{n_L}{\tau_p} + 2^*\frac{\sigma v_g}{V}N_2\left(n_L + \frac{1}{V}\right). \tag{7.20}$$

Experimentally, the photon number and the inversion in a laser resonator are not very convenient quantities, therefore, we normalize both equations to the round-trip amplitude gain $g = 2^* \frac{\sigma v_g}{2V} N_2 T_R$ experienced by the light and the circulating intracavity power $P = I \cdot A_{eff}$

$$\frac{d}{dt} g = -\frac{g - g_0}{\tau_L} - \frac{gP}{E_{sat}} \tag{7.21}$$

$$\frac{d}{dt} P = -\frac{1}{\tau_p} P + \frac{2g}{T_R} \left(P + P_{vac}\right), \tag{7.22}$$

with

$$E_{sat} = \frac{hf_L}{2^* \sigma} A_{eff} = \frac{1}{2^*} I_s A_{eff} \tau_L \tag{7.23}$$

$$P_{sat} = E_s / \tau_L \tag{7.24}$$

$$P_{vac} = hf_L v_g / 2^* L = hf_L / T_R \tag{7.25}$$

$$g_0 = 2^* \frac{R_p}{2A_{eff}} \sigma \tau_L, \tag{7.26}$$

the small signal round-trip gain of the laser. Note, the factor of two in front of gain and loss is due to the fact, that we defined g and l as gain and loss with respect to amplitude. Eq.(7.26) elucidates that the figure of merit that characterizes the small signal gain achievable with a certain laser material is the $\sigma\tau_L$-product.

7.5 Continuous Wave Operation

If $P_{vac} \ll P \ll P_{sat} = E_{sat}/\tau_L$, than $g = g_0$ and we obtain from Eq.(7.22), neglecting P_{vac}

$$\frac{dP}{P} = 2\left(g_0 - l\right) \frac{dt}{T_R} \tag{7.27}$$

or

$$P(t) = P(0) e^{2(g_0 - l)\frac{t}{T_R}}. \tag{7.28}$$

The laser power builts up from vaccum fluctuations, see Figure 7.23 until it reaches the saturation power, when saturation of the gain sets in within the built-up time

$$T_B = \frac{T_R}{2\left(g_0 - l\right)} \ln \frac{P_{sat}}{P_{vac}} = \frac{T_R}{2\left(g_0 - l\right)} \ln \frac{A_{eff} T_R}{\sigma \tau_L}. \tag{7.29}$$

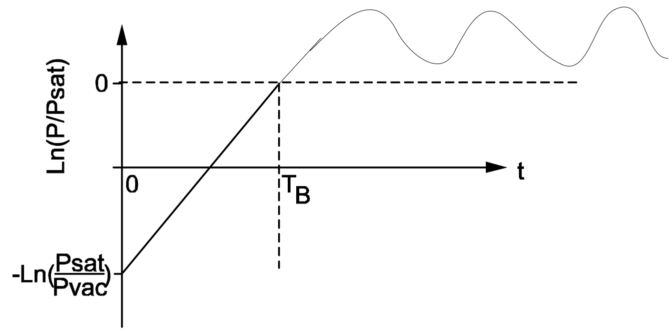

Figure 7.23: Built-up of laser power from spontaneous emission noise.

Some time after the built-up phase the laser reaches steady state, with the saturated gain and steady state power resulting from Eqs.(7.21-7.22), neglecting in the following the spontaneous emission, $P_{vac} = 0$, and for $\frac{d}{dt} = 0$:

$$g_s = \frac{g_0}{1 + \frac{P_s}{P_{sat}}} = l \tag{7.30}$$

$$P_s = P_{sat} \left(\frac{g_0}{l} - 1 \right), \tag{7.31}$$

Figure 7.24 shows output power and gain as a function of small signal gain g_0, which is proportional to the pump rate. Below threshold, the output power is zero and the gain increases linearly with in crease pumping. After reaching threshold the gain stays clamped at the threshold value determined by gain equal loss and the output power increases linearly.

7.6 Stability and Relaxation Oscillations

How does the laser reach steady state, once a perturbation has occured?

$$g = g_s + \Delta g \tag{7.32}$$

$$P = P_s + \Delta P \tag{7.33}$$

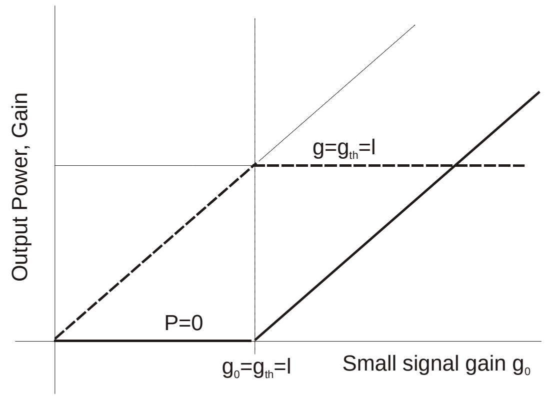

Figure 7.24: Output power and gain of a laser as a function of pump power.

Substitution into Eqs.(7.21-7.22) and linearization leads to

$$\frac{d\Delta P}{dt} = +2\frac{P_s}{T_R}\Delta g \tag{7.34}$$

$$\frac{d\Delta g}{dt} = -\frac{g_s}{E_{sat}}\Delta P - \frac{1}{\tau_{stim}}\Delta g \tag{7.35}$$

where $\frac{1}{\tau_{stim}} = \frac{1}{\tau_L}\left(1 + \frac{P_s}{Psat}\right)$ is the inverse stimulated lifetime. The stimulated lifetime is the lifetime of the upper laser state in the presence of the optical field. The perturbations decay or grow like

$$\begin{pmatrix} \Delta P \\ \Delta g \end{pmatrix} = \begin{pmatrix} \Delta P_0 \\ \Delta g_0 \end{pmatrix} e^{st}. \tag{7.36}$$

which leads to the system of equations (using $g_s = l$)

$$A\begin{pmatrix} \Delta P_0 \\ \Delta g_0 \end{pmatrix} = \begin{pmatrix} -s & 2\frac{P_s}{T_R} \\ -\frac{T_R}{E_{sat}2\tau_p} & -\frac{1}{\tau_{stim}} - s \end{pmatrix}\begin{pmatrix} \Delta P_0 \\ \Delta g_0 \end{pmatrix} = 0. \tag{7.37}$$

There is only a solution, if the determinante of the coefficient matrix vanishes, i.e.

$$s\left(\frac{1}{\tau_{stim}} + s\right) + \frac{P_s}{E_{sat}\tau_p} = 0, \tag{7.38}$$

which determines the relaxation rates or eigen frequencies of the linearized system

$$s_{1/2} = -\frac{1}{2\tau_{stim}} \pm \sqrt{\left(\frac{1}{2\tau_{stim}}\right)^2 - \frac{P_s}{E_{sat}\tau_p}}. \tag{7.39}$$

Introducing the pump parameter $r = 1 + \frac{P_s}{P_{sat}}$, which tells us how often we pump the laser over threshold, the eigen frequencies can be rewritten as

$$s_{1/2} = -\frac{1}{2\tau_{stim}}\left(1 \pm j\sqrt{\frac{4(r-1)}{r}\frac{\tau_{stim}}{\tau_p} - 1}\right), \tag{7.40}$$

$$= -\frac{r}{2\tau_L} \pm j\sqrt{\frac{(r-1)}{\tau_L\tau_p} - \left(\frac{r}{2\tau_L}\right)^2} \tag{7.41}$$

There are several conclusions to draw:

- (i): The stationary state $(0, g_0)$ for $g_0 < l$ and (P_s, g_s) for $g_0 > l$ are always stable, i.e. $\mathrm{Re}\{s_i\} < 0$.
- (ii): For lasers pumped above threshold, $r > 1$, and long upper state lifetimes, i.e. $\frac{r}{4\tau_L} < \frac{1}{\tau_p}$,

  the relaxation rate becomes complex, i.e. there are relaxation oscillations

  $$s_{1/2} = -\frac{1}{2\tau_{stim}} \pm j\omega_R. \tag{7.42}$$

  with a frequency ω_R approximately equal to the geometric mean of inverse stimulated lifetime and photon life time

  $$\omega_R \approx \sqrt{\frac{1}{\tau_{stim}\tau_p}}. \tag{7.43}$$

- If the laser can be pumped strong enough, i.e. r can be made large enough so that the stimulated lifetime becomes as short as the cavity decay time, relaxation oscillations vanish.

The physical reason for relaxation oscillations and instabilities related to it is, that the gain reacts to slow on the light field, i.e. the stimulated lifetime is long in comparison with the cavity decay time.

Example: diode-pumped Nd:YAG-Laser

$$\lambda_0 = 1064 \text{ nm}, \sigma = 4\cdot 10^{-20} cm^2, \; A_{eff} = \pi\,(100\mu m \times 150\mu m)\,, r = 50$$
$$\tau_L = 1.2\; ms,\, l = 1\%,\, T_R = 10ns$$

From Eq.(7.16) we obtain:

$$\begin{aligned} I_{sat} &= \frac{hf_L}{\sigma\tau_L} = 3.9\frac{kW}{cm^2},\ P_{sat} = I_{sat}A_{eff} = 1.8\ W,\ P_s = 91.5W \\ \tau_{stim} &= \frac{\tau_L}{r} = 24\mu s,\ \tau_p = 1\mu s, \omega_R = \sqrt{\frac{1}{\tau_{stim}\tau_p}} = 2\cdot 10^5 s^{-1}. \end{aligned}$$

Figure 7.25 shows the typically observed fluctuations of the output of a solid-

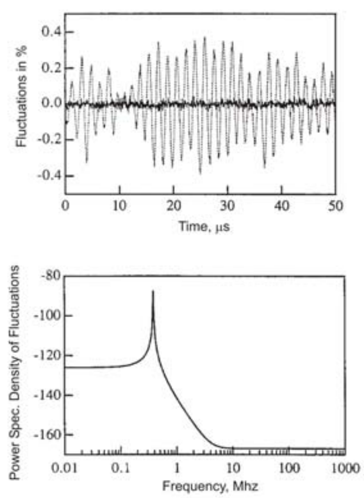

Figure 7.25: Relaxation oscillations in the time and frequency domain.

state laser in the time and frequency domain. Note, that this laser has a long upperstate lifetime of several 100 μs

One can also define a quality factor for the relaxation oscillations by the ratio of the imaginary to the real part of the complex eigen frequencies 7.41

$$Q = \sqrt{\frac{4\tau_L}{\tau_p}\frac{(r-1)}{r^2}}. \tag{7.44}$$

The quality factor can be as large a several thousand for solid-state lasers with long upper-state lifetimes in the millisecond range.

7.7 Laser Efficiency

An important measure for a laser is the efficiency with which pump power is converted into laser output power. To determine the efficiency we must review the important parameters of a laser and the limitations these parameters impose.

From Eq.(7.31) we found that the steady state intracavity power P_s of a laser is

$$P_s = P_{sat}\left(\frac{2g_0}{2l} - 1\right), \tag{7.45}$$

where $2g_0$ is the small signal round-trip power gain, P_{sat} the gain saturation power and $2l$ is the power loss per round-trip. Both parameters are expressed in Eqs.(7.23)-(7.26) in terms of the fundamental pump parameter R_p, $\sigma\tau_L$-product and mode cross section A_{eff} of the gain medium. For this derivation it was asummed that all pumped atoms are in the laser mode with constant intensity over the beam cross section

$$2g_0 = 2^* \frac{R_p}{A_{eff}} \sigma\tau_L, \tag{7.46}$$

$$P_{sat} = \frac{hf_L}{2^*\sigma\tau_L} A_{eff} \tag{7.47}$$

The power losses of lasers are due to the internal losses $2l_{int}$ and the transmission T through the output coupling mirror. The internal losses can be a significant fraction of the total losses. The output power of the laser is

$$P_{out} = T \cdot P_{sat}\left(\frac{2g_0}{2l_{int} + T} - 1\right) \tag{7.48}$$

The pump power of a laser is minimized given

$$P_p = R_p h f_P, \tag{7.49}$$

where hf_P is the energy of the pump photons. In discussing the efficiency of a laser, we consider the overall efficiency

$$\eta = \frac{P_{out}}{P_p} \tag{7.50}$$

which approaches the differential efficiency η_D if the laser is pumped many times over threshold, i.e. $r = 2g_0/2l \to \infty$

$$\begin{aligned}
\eta_D &= \frac{\partial P_{out}}{\partial P_p} = \eta(r \to \infty) && (7.51)\\
&= \frac{T}{2l_{int} + T} P_{sat} \frac{2^*}{A_{eff} h f_P} \sigma \tau_L && (7.52)\\
&= \frac{T}{2l_{int} + T} \cdot \frac{h f_L}{h f_P}. && (7.53)
\end{aligned}$$

Thus the efficiency of a laser is fundamentally limited by the ratio of output coupling to total losses and the quantum defect in pumping. Therefore, one would expect that the optimum output coupling is achieved with the largest output coupler, however, this is not true as we considered the case of operating many times above threshold.

7.8 "Thresholdless" Lasing

So far we neglected the spontaneous emission into the laser mode. This is justified for large lasers where the density of radiation modes in the laser medium is essentially the free space mode density and effects very close to threshold are not of interest. For lasers with small mode volume, or a laser operating very close to threshold, the spontaneous emission into the laser mode can no longer be neglected and we should use the full rate equations (7.21) and (7.22)

$$\begin{aligned}
\frac{d}{dt} g &= -\frac{g - g_0}{\tau_L} - \frac{gP}{E_{sat}} && (7.54)\\
\frac{d}{dt} P &= -\frac{1}{\tau_p} P + \frac{2g}{T_R} \left(P + P_{vac}\right), && (7.55)
\end{aligned}$$

where P_{vac} is the power of a single photon in the mode. The steady state conditions are

$$\begin{aligned}
g_s &= \frac{g_0}{(1 + P_s/P_{sat})}, && (7.56)\\
0 &= (2g_s - 2l)\, P + 2g_s P_{vac}. && (7.57)
\end{aligned}$$

Substitution of the saturated gain condition (7.56) into (7.57) and using the pump parameter $r = 2g_0/2l$, leads to a quadratic equation for the normalized intracavity steady state power p=P_s/P_{sat} in terms of normalized vacuum power $p_v = P_{vac}/P_{sat} = \sigma\tau_L v_g/V$. This equation has the solutions

$$p = \frac{r - 1 + rp_v}{2} \pm \sqrt{\left(\frac{r - 1 + rp_v}{2}\right)^2 + (rp_v)^2}. \tag{7.58}$$

where only the solution with the plus sign is of physical significance. Note, the typical value for the $\sigma\tau_L$-product of the laser materials in table 7.1 is $\sigma\tau_L = 10^{-23}cm^2s$. If the volume is measured in units of wavelength cubed we obtain $p_v = 0.3/\beta$ for$\lambda = 1\mu m, V = \beta\lambda^3$ and $v_g = c$. Figure 7.26 shows the behavior of the intracavity power as a function of the pump parameter for various values of the normalized vacuum power.

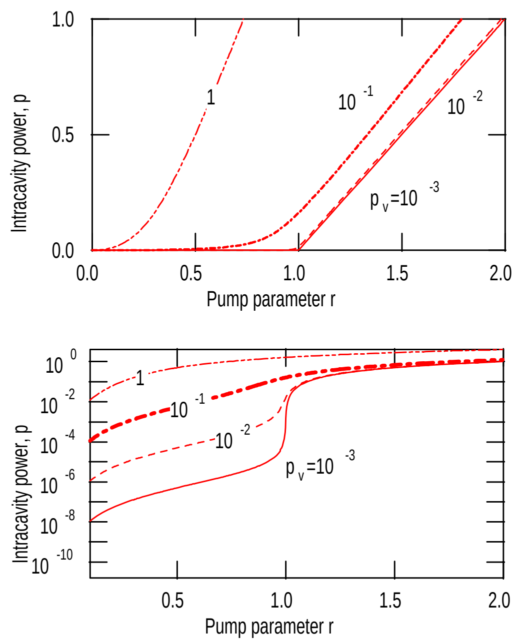

Figure 7.26: Intracavity power as a function of pump parameter r on a linear scale (a) and a logarithmic scale (b) for various values of the normalized vacuum power.