6.4 The Density Operator

To study incoherent or dissipative processes it is necessary to switch to a statistical description. That is, we investigate not only the interaction of the atoms with the light field, via the Schroedinger Equation, leading to Rabi-oscillations but rather the interaction of an atomic ensemble with the light field. This is achieved by using the density operator instead of deterministic wave functions, similar to classical statistical mechanics, where the deterministic trajectories of particles are replaced by probability distributions.

The density operator of a pure state is defined by the dyadic product of the state with itself

$$\boldsymbol{\rho} = |\psi\rangle \langle\psi| \tag{6.58}$$

or in the energy representation by a $2 \times 2-$matrix

$$\boldsymbol{\rho} = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix}. \tag{6.59}$$

. In the case of a pure state (6.9) this is

$$\boldsymbol{\rho} = \begin{pmatrix} c_e c_e^* & c_e c_g^* \\ c_g c_e^* & c_g c_g^* \end{pmatrix}. \tag{6.60}$$

For the rather simple case of a two-level system, each element of the density matrix corresponds to a physical quantity. The main diagonal contains the population probabilities for the levels; the off-diagonal element is the expectation value of the positive or negative frequency component of the dipole moment of the atom, i.e. its contribution to the polarization in the medium.

However, the concept of a density operator can be applied to any quantum mechanical system, not just the two-level atom. If an ensemble is described by a density operator, the expectation value of an arbitrary operator $\mathbf{A}$ can be computed using the trace formula

$$\langle \mathbf{A} \rangle = Tr\{\boldsymbol{\rho}\mathbf{A}\,\}. \tag{6.61}$$

The trace of an operator is defined as

$$Tr\{\mathbf{O}\,\} = \sum_n \langle n| \mathbf{O} |n\rangle. \tag{6.62}$$

where $|n\rangle$ can be any complete orthonormal base (ONB) in the Hilbert space. For example for the density matrix of the pure state (6.58) we find

$$\langle \mathbf{A} \rangle \;=\; Tr\{\boldsymbol{\rho}\mathbf{A}\,\} = \sum_n \langle n| \, \boldsymbol{\rho}\mathbf{A} \, |n\rangle \tag{6.63}$$

$$=\; \sum_n \langle n \, |\psi\rangle \, \langle \psi| \, \mathbf{A} \, |n\rangle = \langle \psi| \, \mathbf{A} \sum_n |n\rangle \, \langle n \, |\psi\rangle \tag{6.64}$$

$$=\; \langle \psi| \, \mathbf{A} \, |\psi\rangle \,. \tag{6.65}$$

The advantage of the density operator is that it can also be applied to a statistical mixture of pure states. For example, if the atom is in state $|e\rangle$ with probability p_e and in state $||g\rangle$ with probability p_g a density operator

$$\boldsymbol{\rho} = p_e \, |e\rangle \, \langle e| + p_g \, |g\rangle \, \langle g| \tag{6.66}$$

is defined, which can be used to compute the average values of observables in the proper statistical sense

$$\langle \mathbf{A} \rangle = Tr\{\boldsymbol{\rho}\mathbf{A}\} = p_e \, \langle e| \, \mathbf{A} \, |e\rangle + p_g \, \langle g| \, \mathbf{A} \, |g\rangle \,. \tag{6.67}$$

Since the matrices (6.27) to (6.30) build a complete base in the space of $2 \times 2-$matrices, we can express the density matrix as

$$\boldsymbol{\rho} \;=\; \rho_{ee}\frac{1}{2}(\mathbf{1} + \boldsymbol{\sigma}_z) + \rho_{gg}\frac{1}{2}(\mathbf{1} - \boldsymbol{\sigma}_z) + \rho_{eg}\boldsymbol{\sigma}^+ + \rho_{ge}\boldsymbol{\sigma}^- \tag{6.68}$$

$$=\; \frac{1}{2}\mathbf{1} + \frac{1}{2}(\rho_{ee} - \rho_{gg})\boldsymbol{\sigma}_z + \rho_{eg}\boldsymbol{\sigma}^+ + \rho_{ge}\boldsymbol{\sigma}^- , \tag{6.69}$$

since the trace of the density matrix is always one (normalization). Choosing the new base $\mathbf{1}, \boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z$, we obtain

$$\boldsymbol{\rho} = \frac{1}{2}\mathbf{1} + \frac{1}{2}(\rho_{ee} - \rho_{gg})\boldsymbol{\sigma}_z + d_x\boldsymbol{\sigma}_x + d_y\boldsymbol{\sigma}_y, \tag{6.70}$$

with

$$d_x \;=\; \frac{1}{2}\left(\rho_{eg} + \rho_{ge}\right) = \Re\{\langle \boldsymbol{\sigma}^{(+)}\rangle\}, \tag{6.71}$$

$$d_y \;=\; \frac{\mathrm{j}}{2}\left(\rho_{eg} - \rho_{ge}\right) = \Im\{\langle \boldsymbol{\sigma}^{(+)}\rangle\}. \tag{6.72}$$

The expectation value of the dipole operator is given by (6.36)

$$\left\langle \vec{\mathbf{d}} \right\rangle = Tr\{\boldsymbol{\rho}\vec{\mathbf{d}}\} = -\vec{M}^* Tr\{\boldsymbol{\rho}\,\boldsymbol{\sigma}^+\} + c.c. = -\vec{M}^* \rho_{ge} + c.c. \tag{6.73}$$

From the Schrödinger equation for the wave function $|\psi>$ we can easily derive the equation of motion for the density operator called the von Neumann equation

$$\dot{\boldsymbol{\rho}} = \frac{d}{dt}|\psi\rangle\langle\psi| + h.c. = \frac{1}{j\hbar}\mathbf{H}|\psi\rangle\langle\psi| - \frac{1}{j\hbar}|\psi\rangle\langle\psi|\mathbf{H} \qquad (6.74)$$
$$= \frac{1}{j\hbar}[\mathbf{H}, \boldsymbol{\rho}].$$

Due to the linear nature of this equation, this is also the correct equation for a density operator describing an arbitrary mixture of states. In case of a two-level atom, the von Neumann equation is

$$\dot{\boldsymbol{\rho}} = \frac{1}{j\hbar}[\mathbf{H}_A, \boldsymbol{\rho}] = -j\frac{\omega_{\in g}}{2}[\boldsymbol{\sigma}_z, \boldsymbol{\rho}]. \qquad (6.75)$$

Using the commutator relations (6.16) - (6.18), the result is

$$\dot{\rho}_{\in e} = 0, \qquad (6.76)$$
$$\dot{\rho}_{gg} = 0, \qquad (6.77)$$
$$\dot{\rho}_{eg} = -j\omega_{eg}\rho_{eg} \quad \rightarrow \quad \rho_{eg}(t) = e^{-j\omega_{eg}t}\rho_{eg}(0), \qquad (6.78)$$
$$\dot{\rho}_{ge} = j\omega_{eg}\rho_{ge} \quad \rightarrow \quad \rho_{ge}(t) = e^{j\omega_{eg}t}\rho_{ge}(0). \qquad (6.79)$$

Again the isolated two-level atom has rather simple dynamics. The populations are constant. If there is a dipole moment induced at $t = 0$, i.e. the system is in a superposition state, then this dipole moment oscillates with the transition frequency $\omega_{\in g}$.

6.5   Energy- and Phase-Relaxation

In reality one has to work very hard to isolated an atom from its environment. Indeed in the case of laser active media, we are interested at radiating atoms, i.e. atoms that have a dipole interaction with the field. The coupling with the infinitely many modes of the free field leads already to spontaneous emission, an irreversible process. We could treat this process by using the Hamiltonian

$$\mathbf{H} = \mathbf{H}_A + \mathbf{H}_F + \mathbf{H}_{A-F}. \qquad (6.80)$$

Here, $\mathbf{H}_A$ is the Hamiltonian of the atom, $\mathbf{H}_F$ of the free field in thermal equilibrium at temperature T, and $\mathbf{H}_{A-F}$ describes the interaction between

them. A complete treatment along these lines would be straight forward using the techniques we learned so far, however it is beyond the scope of this class. The result of this calculation leads to the von Neumann equation of the reduced density matrix, i.e. the density matrix of the atom alone. In fact the result of such a calculation gives for the diagonal elements of the density operator, i.e. the state population probabilities, equations identical to those in section 3.3 involving Einstein's A and B coefficients. With the spontaneous emission rate $A = 1/\tau_{sp}$, i.e. the inverse spontaneous life time τ_{sp}, the populations change due to the induced and spontaneious emission processes and the absorption processes

$$\frac{d}{dt}|c_e(t)|^2 = \frac{d}{dt}\rho_{ee} = -\Gamma_e \rho_{ee} + \Gamma_a \rho_{gg} \tag{6.81}$$

with the abbreviations

$$\Gamma_e = \frac{1}{\tau_{sp}}(n_{th} + 1), \tag{6.82}$$

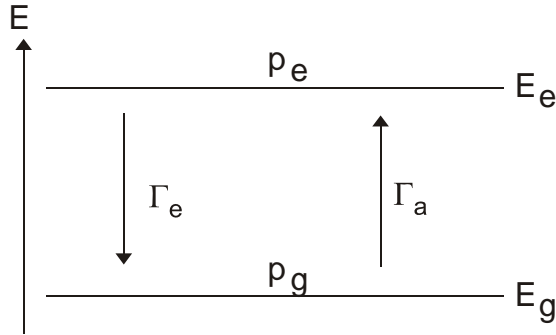$$\Gamma_a = \frac{1}{\tau_{sp}}n_{th}. \tag{6.83}$$

see Figure 6.3.

Figure 6.3: Two-level atom with transistion rates due to induced and spontaneous emission and absorption.

Here n_{th} is the number of thermally excited photons in the modes of the free field with frequency ω_{eg}, $n_{th} = 1/(\exp(\hbar\omega_{eg}/kT) - 1)$, at temperature T.

The total probability of being in the excited or the ground state has to be maintained, that is

$$\frac{d}{dt}\rho_{gg} = -\frac{d}{dt}\rho_{ee} = \Gamma_e\rho_{ee} - \Gamma_a\rho_{gg}. \tag{6.84}$$

If the populations decay, the polarization does as well, since $\rho_{ge} = c_e^* c_g$. It turns out that the polarization dynamics according to Eq.(6.78), besides the coherent oscillation, also aquires a decay process due to the finite lifetime of the excited state

$$\frac{d}{dt}\rho_{ge} = j\omega_{eg}\rho_{eg} - \frac{\Gamma_e + \Gamma_a}{2}\rho_{ge}. \tag{6.85}$$

Thus the absorption as well as the emission processes are destructive to the phase. Therefore, the corresponding rates add up in the phase decay rate.

Taking the coherent (6.76)-(6.79) and incoherent processes (6.84-6.85) into account results in the following equations for the normalized average dipole moment $d = d_x + jd_y$ and the inversion w

$$\dot{d} = \dot{\rho}_{ge} = (j\omega_{eg} - \frac{1}{T_2})d, \tag{6.86}$$

$$\dot{w} = \dot{\rho}_{ee} - \dot{\rho}_{gg} = -\frac{w - w_0}{T_1}, \tag{6.87}$$

with the time constants

$$\frac{1}{T_1} = \frac{2}{T_2} = \Gamma_e + \Gamma_a = \frac{2n_{th} + 1}{\tau_{sp}} \tag{6.88}$$

and the equilibrium inversion w_0, due to the thermal excitation of the atom by the thermal field

$$w_0 = \frac{\Gamma_a - \Gamma_e}{\Gamma_a + \Gamma_e} = \frac{-1}{1 + 2n_{th}} = -\tanh\left(\frac{\hbar\omega_{eg}}{2kT}\right). \tag{6.89}$$

The time constant T_1 denotes the energy relaxation in the two-level system and T_2 the phase relaxation. T_2 is the correlation time between amplitudes c_e and c_g. The coherence between the excited and the ground state described by ρ_{ge} is destroyed by the interaction of the two -level system with the environment.

In this basic model, the energy relaxation is half the phase relaxation rate or

$$T_2 = 2T_1. \tag{6.90}$$

The atoms in a laser medium do not only interact with the electromagnetic field, but also with phonons, i.e. acoustic vibrations of the host lattice in solid state laser material. Atoms might collide with each other in a gas laser and so on. All these processes must be considered when determining the energy and phase relaxation rates. Thus it might be not only radiative transistions that lead to a finite energy relaxation time T_1. Some of the processes are elastic, i.e. there is no energy relaxation but only the phase is influenced during the collision. Therefore, these processes reduce T_2 but have no influence on T_1. In real systems the phase relaxation time is most often much shorter than twice the energy relaxation time.

$$T_2 \leq 2T_1. \tag{6.91}$$

If the inversion deviates from its equilibrium value, w_0, it relaxes back into equilibrium with a time constant T_1. Eq. (6.89) shows that for all temperatures $T > 0$ the inversion is negative, i.e. the lower level is stronger populated than the upper level. Thus with incoherent thermal light, inversion in a two-level system cannot be achieved. Inversion can only be achieved by pumping with incoherent light, if there are more levels and subsequent relaxation processes into the upper laser level. Due to these relaxation processes the rate Γ_a deviates from the equilibrium expression (6.83), and it has to be replaced by the pump rate Λ. If the pump rate Λ exceeds Γ_e, the inversion corresponding to Eq. (6.89) becomes positive,

$$w_0 = \frac{\Lambda - \Gamma_e}{\Lambda + \Gamma_e}. \tag{6.92}$$

If we allow for artificial negative temperatures, we obtain with $T < 0$ for the ratio of relaxation rates

$$\frac{\Gamma_e}{\Gamma_a} = \frac{1 + \bar{n}}{\bar{n}} = e^{\frac{\hbar \omega_{eg}}{kT}} < 1. \tag{6.93}$$

Thus the pumping of the two-level system drives the system far away from thermal equilibrium. Now, we have a correct description of an ensemble of atoms in thermal equilibrium with its environment, which is a much more realistic description of media especially of typical laser media.

6.6   The Bloch Equations

If there is a coherent additional field in addition to the coupling to the environment, the Hamiltonian has to be extended by the dipole interaction with

that field,

$$\mathbf{H}_E = -\mathbf{d} \cdot \vec{E}(\vec{x}_A, t). \tag{6.94}$$

Again we use the interaction Hamiltonian in RWA according to Eq.(6.40) for a time harmonic field Eq.(6.38) with polarization vector $\vec{e}_p$

$$\mathbf{H}_E = \frac{1}{2} \left(\vec{M} \cdot \vec{e}_p^* \right) \underline{E}_0^* e^{-j\omega t} \boldsymbol{\sigma}^+ + h.c.. \tag{6.95}$$

In the von Neumann equation this leads to the additional term

$$\dot{\boldsymbol{\rho}}|_E = \frac{1}{j\hbar}[\mathbf{H}_E, \boldsymbol{\rho}] \tag{6.96}$$

$$= -j\Omega_r e^{-j\omega t}[\boldsymbol{\sigma}^+, \boldsymbol{\rho}] + j\Omega_r^* e^{j\omega t}[\boldsymbol{\sigma}^-, \boldsymbol{\rho}]. \tag{6.97}$$

With the density operator expressed as

$$\boldsymbol{\rho} = \frac{1}{2}\mathbf{1} + \frac{1}{2}(\rho_{ee} - \rho_{gg})\boldsymbol{\sigma}_z + \rho_{eg}\boldsymbol{\sigma}^+ + \rho_{ge}\boldsymbol{\sigma}^-, \tag{6.98}$$

and the commutation relations (6.16) - (6.18) we find

$$\dot{\boldsymbol{\rho}}|_E = \frac{1}{2}(\dot{\rho}_{ee} - \dot{\rho}_{gg})\boldsymbol{\sigma}_z + \dot{\rho}_{eg}\boldsymbol{\sigma}^+ + \dot{\rho}_{ge}\boldsymbol{\sigma}^- \tag{6.99}$$

$$= -j\Omega_r e^{-j\omega t} \left\{ \frac{1}{2}(\rho_{ee} - \rho_{gg}) \left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}_z \right] + \rho_{ge} \left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^- \right] \right\} +$$

$$+ j\Omega_r^* e^{j\omega t} \left[\left\{ \frac{1}{2}(\rho_{ee} - \rho_{gg}) \left[\boldsymbol{\sigma}^-, \boldsymbol{\sigma}_z \right] + \rho_{eg} \left[\boldsymbol{\sigma}^-, \boldsymbol{\sigma}^+ \right] \right\} \right.$$

$$= j\Omega_r e^{-j\omega t} \left\{ (\rho_{ee} - \rho_{gg})\boldsymbol{\sigma}^+ + \rho_{ge}\boldsymbol{\sigma}_z \right\}$$

$$+ j\Omega_r^* e^{j\omega t} \left[\left\{ (\rho_{ee} - \rho_{gg})\boldsymbol{\sigma}^- - \rho_{eg}\boldsymbol{\sigma}_z \right\} \right.$$

or expressed by the components of the density operator

$$(\dot{\rho}_{ee} - \dot{\rho}_{gg})|_E = 2j\Omega_r e^{-j\omega t}\rho_{ge} + c.c., \tag{6.100}$$

$$\dot{\rho}_{ge}|_E = j\Omega_r^* e^{j\omega t}(\rho_{ee} - \rho_{gg}), . \tag{6.101}$$

The interaction with the external field leads to the following contributions in the dynamics of the dipole moment and the inversion

$$\dot{d}|_E = \dot{\rho}_{ge}|_E = j\Omega_r^* e^{j\omega t} w, \tag{6.102}$$

$$\dot{w}|_E = \dot{\rho}_{ee}|_E - \dot{\rho}_{gg}|_E = 2j\Omega_r e^{-j\omega t} d + c.c. \tag{6.103}$$

Thus, the total dynamics of the two-level system including the pumping and dephasing processes from Eqs.(6.86) and (6.87) is given by

$$\dot{d} \; = \; -(\frac{1}{T_2} - \mathrm{j}\omega_{eg})d + \mathrm{j}\Omega_r^* e^{\mathrm{j}\omega t}\, w, \qquad\qquad (6.104)$$

$$\dot{w} \; = \; -\frac{w - w_0}{T_1} + 2\mathrm{j}\Omega_r e^{-\mathrm{j}\omega t}d - 2\mathrm{j}\Omega_r^* e^{\mathrm{j}\omega t}d^*. \qquad (6.105)$$

These equations are called the Bloch Equations. They describe the dynamics of a statistical ensemble of two-level atoms interacting with a classical electric field. Together with the Maxwell-Equations, where the polarization of the medium is related to the expectation value of the dipole moment of the atomic ensemble these result in the Maxwell-Bloch Equations.

6.7   Dielectric Susceptibility and Saturation

We have assumed that the external field is time harmonic

$$\vec{E}(\vec{x}_A, t) = \frac{1}{2}\left(\underline{E}_0 e^{\mathrm{j}\omega t}\, \vec{e}_p + \underline{E}_0^* e^{-\mathrm{j}\omega t}\, \vec{e}_p^*\right). \qquad (6.106)$$

The Bloch Equations are nonlinear. However, for moderate field intensities, i.e. the magnitude of the Rabi-frequency is much smaller than the optical frequency, $|\Omega_r| << \omega$, the inversion does not change much within an optical cycle of the field. We assume that the inversion w of the atom will only be slowly changing and it adjusts itself to a steady state value w_s. If the inversion can be assumed time independent, $w = w_s$ the equation for the dipole moment is linear and the dipole moment will oscillate with the same frequency as the driving field

$$\underline{d} = \underline{d}_0 e^{\mathrm{j}\omega t}. \qquad\qquad (6.107)$$

With the time harmonic solution (6.107) we find from Eqs. (6.104) and (6.105) for the dipole amplitude and the steady state inversion

$$\underline{d}_0 \; = \; \frac{\mathrm{j}}{2\hbar}\frac{\left(\vec{M}^* \cdot \vec{e}_p\right) w_s}{1/T_2 + \mathrm{j}(\omega - \omega_{eg})}\underline{E}_0 \qquad\qquad (6.108)$$

$$w_s \; = \; \frac{w_0}{1 + \frac{T_1}{\hbar^2}\frac{1/T_2\, |\vec{M}^* \cdot \vec{e}_p|^2}{(1/T_2)^2 + (\omega_{eg} - \omega)^2}|\underline{E}_0|^2}. \qquad (6.109)$$

We introduce the normalized lineshape function, which is in this case a Lorentzian,

$$L(\omega) = \frac{(1/T_2)^2}{(1/T_2)^2 + (\omega_{eg} - \omega)^2}, \tag{6.110}$$

and connect the square of the field $|\underline{E}_0|^2$ to the intensity I of a propagating plane wave, according to Eq. (2.38), $I = \frac{1}{2Z_F}|\underline{E}_0|^2$,

$$w_s = \frac{w_0}{1 + \frac{I}{I_s}L(\omega)}. \tag{6.111}$$

Thus the stationary inversion depends on the intensity of the incident light. Therefore, w_0 is called the unsaturated inversion, w_s the saturated inversion and I_s, with

$$I_s = \left[\frac{2T_1 T_2 Z_F}{\hbar^2}|\vec{M}^* \cdot \vec{e}_p|^2\right]^{-1}, \tag{6.112}$$

is the saturation intensity. The expectation value of the dipole operator (6.31) is then given by

$$\left\langle \tilde{\mathbf{d}} \right\rangle = -(\vec{M}^*\underline{d} + \vec{M}\underline{d}^*). \tag{6.113}$$

Multiplication with the number of atoms per unit volume, N, relates the dipole moment of the atom to the complex polarization $\underline{\vec{P}}_0$ of the medium, and therefore to the susceptibility according to

$$\underline{\vec{P}}_0 = -2N\vec{M}^*\underline{d}_0, \tag{6.114}$$

$$\underline{\vec{P}}_0 = \epsilon_0 \chi(\omega)\vec{e}_p\underline{E}_0. \tag{6.115}$$

From the definitions (6.114), (6.115) and Eq. (6.108) we obtain for the linear susceptibility of the medium

$$\chi(\omega) = \vec{M}^*\vec{M}^T \frac{\mathrm{j}N}{\hbar\epsilon_0}\frac{w_s}{1/T_2 + \mathrm{j}(\omega - \omega_{eg})}, \tag{6.116}$$

which is a tensor. In the following we assume that the direction of the atom is random, i.e. the alignment of the atomic dipole moment, $\vec{M}$, and the electric field is random. Therefore, we have to average over the angle enclosed

between the electric field of the wave and the atomic dipole moment, which results in

$$\overline{\begin{pmatrix} M_x M_x & M_x M_y & M_x M_z \\ M_y M_x & M_y M_y & M_y M_z \\ M_z M_x & M_z M_y & M_z M_z \end{pmatrix}} = \begin{pmatrix} \overline{M_x^2} & 0 & 0 \\ 0 & \overline{M_y^2} & 0 \\ 0 & 0 & \overline{M_z^2} \end{pmatrix} = \frac{1}{3}|\vec{M}|^2\, \mathbf{1}. \quad (6.117)$$

Thus, for homogeneous and isotropic media the susceptibility tensor shrinks to a scalar

$$\chi(\omega) = \frac{1}{3}|\vec{M}|^2 \frac{jN}{\hbar\epsilon_0} \frac{w_s}{1/T_2 + j(\omega - \omega_{eg})}. \tag{6.118}$$

Real and imaginary part of the susceptibility

$$\chi(\omega) = \chi'(\omega) + j\chi''(\omega) \tag{6.119}$$

are then given by

$$\chi'(\omega) = -\frac{|\vec{M}|^2 N w_s T_2^2 (\omega_{eg} - \omega)}{3\hbar\epsilon_0} L(\omega), \tag{6.120}$$

$$\chi''(\omega) = \frac{|\vec{M}|^2 N w_s T_2}{3\hbar\epsilon_0} L(\omega). \tag{6.121}$$

If the incident radiation is weak, i.e.

$$\frac{I}{I_s} L(\omega)) \ll 1 \tag{6.122}$$

we obtain $w_s \approx w_0$. For optical transitions there is no thermal excitation of the excited state and $w_0 = -1$. For an inverted system, $w_0 > 0$, the real and imaginary parts of the susceptibility are shown in Fig. 6.4.

The shape of the susceptibility computed quantum mechanically compares well with the classical susceptibility (2.43) derived from the harmonic oscillator model close to the transistion frequency ω_{eg} for a transition with reasonably high $Q = T_2\omega_{eg}$. Note, the quantum mechanical susceptibility is identical to the complex Lorentzian introduced in Eq.(2.90). There is an appreciable deviation, however, far away from resonance. Far off resonance the rotating wave approximation should not be used.

The physical meaning of the real and imaginary part of the susceptibility is of course identical to section 2.1.8. The propagation constant k of a TEM-wave in such a medium is related to the susceptibility by

$$k = \omega\sqrt{\mu_0\epsilon_0(1 + \chi(\omega))} \approx k_0\left(1 + \frac{1}{2}\chi(\omega)\right), \quad \text{with} \quad k_0 = \omega\sqrt{\mu_0\epsilon_0} \tag{6.123}$$

for $|\chi| \ll 1$. Under this assumption we obtain

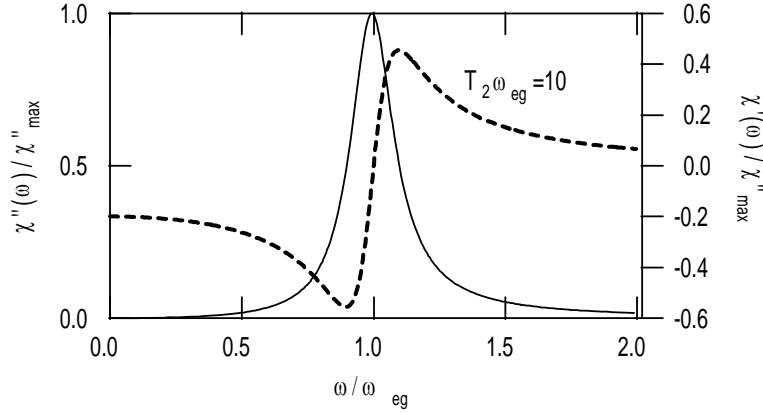$$k = k_0(1 + \frac{\chi'}{2}) + \mathrm{j}k_0\frac{\chi''}{2}. \tag{6.124}$$

Figure 6.4: Real and imaginary part of the complex susceptibility for an inverted medium $w_s > 0$. The positive imaginary susceptibility indicates exponential growth of an electromagnetic wave propagating in the medium.

The real part of the susceptibility contributes to the refractive index $n = 1 + \chi'/2$. In the case of $\chi'' < 0$, the imaginary part leads to an exponential damping of the wave. For $\chi'' > 0$ amplification takes place. Amplification of the wave is possible for $w_0 > 0$, i.e. an inverted medium.

The phase relaxation rate $1/T_2$ of the dipole moment determines the width of the absorption line or the bandwidth of the amplifier. The amplification can not occur forever, because the amplifier saturates when the intensity reaches the saturation intensity. This is a strong deviation from the linear susceptibility we derived from the classical oscillator model. The reason for this saturation is two fold. First, the light can not extract more energy from the atoms then there is energy stored in them, i.e. energy conservation holds. Second the induced dipole moment in a two-level atom is limited by the maximum value of the matrix element. In contrast the induced dipole moment in a classical oscillator grows proportionally to the applied field without limits.