Chapter 2

Classical Electromagnetism and Optics

The classical electromagnetic phenomena are completely described by Maxwell's Equations. The simplest case we may consider is that of electrodynamics of isotropic media

2.1 Maxwell's Equations of Isotropic Media

Maxwell's Equations are

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}, \tag{2.1a}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{2.1b}$$

$$\nabla \cdot \vec{D} = \rho, \tag{2.1c}$$

$$\nabla \cdot \vec{B} = 0. \tag{2.1d}$$

The material equations accompanying Maxwell's equations are:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}, \tag{2.2a}$$

$$\vec{B} = \mu_0 \vec{H} + \vec{M}. \tag{2.2b}$$

Here, $\vec{E}$ and $\vec{H}$ are the electric and magnetic field, $\vec{D}$ the dielectric flux, $\vec{B}$ the magnetic flux, $\vec{J}$ the current density of free chareges, ρ is the free charge density, $\vec{P}$ is the polarization, and $\vec{M}$ the magnetization.

Note, it is Eqs.(2.2a) and (2.2b) which make electromagnetism an interesting and always a hot topic with never ending possibilities. All advances in engineering of artifical materials or finding of new material properties, such as superconductivity, bring new life, meaning and possibilities into this field.

By taking the curl of Eq. (2.1b) and considering

$$\nabla \times \left(\nabla \times \vec{E} \right) = \nabla \left(\nabla \cdot \vec{E} \right) - \Delta \vec{E},$$

where ∇ is the Nabla operator and Δ the Laplace operator, we obtain

$$\Delta \vec{E} - \mu_0 \frac{\partial}{\partial t} \left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t} \right) = \frac{\partial}{\partial t} \nabla \times \vec{M} + \nabla \left(\nabla \cdot \vec{E} \right) \qquad (2.3)$$

and hence

$$\left(\Delta - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) \vec{E} = \mu_0 \left(\frac{\partial \vec{j}}{\partial t} + \frac{\partial^2}{\partial t^2} \vec{P} \right) + \frac{\partial}{\partial t} \nabla \times \vec{M} + \nabla \left(\nabla \cdot \vec{E} \right). \qquad (2.4)$$

with the vacuum velocity of light

$$c_0 = \sqrt{\frac{1}{\mu_0 \epsilon_0}}. \qquad (2.5)$$

For dielectric non magnetic media, which we often encounter in optics, with no free charges and currents due to free charges, there is $\vec{M} = \vec{0}, \ \vec{J} = \vec{0}, \ \rho = 0$, which greatly simplifies the wave equation to

$$\left(\Delta - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) \vec{E} = \mu_0 \frac{\partial^2}{\partial t^2} \vec{P} + \nabla \left(\nabla \cdot \vec{E} \right). \qquad (2.6)$$

2.1.1   Helmholtz Equation

In general, the polarization in dielectric media may have a nonlinear and non local dependence on the field. For linear media the polarizability of the medium is described by a dielectric susceptibility $\chi(\vec{r}, t)$

$$\vec{P}(r, t) = \epsilon_0 \int \int d\vec{r}' dt' \ \chi(\vec{r} - \vec{r}', t - t') \vec{E}(\vec{r}', t'). \qquad (2.7)$$

The polarization in media with a local dielectric suszeptibility can be described by

$$\vec{P}(r,t) = \epsilon_0 \int dt' \, \chi\left(\vec{r}, t-t'\right) \vec{E}\left(\vec{r}, t'\right). \tag{2.8}$$

This relationship further simplifies for homogeneous media, where the susceptibility does not depend on location

$$\vec{P}(\vec{r},t) = \epsilon_0 \int dt' \, \chi\left(t-t'\right) \vec{E}\left(\vec{r}, t'\right). \tag{2.9}$$

which leads to a dielectric response function or permittivity

$$\epsilon\left(t\right) = \epsilon_0(\delta(t) + \chi\left(t\right)) \tag{2.10}$$

and with it to

$$\vec{D}(r,t) = \int dt' \, \epsilon\left(t-t'\right) \vec{E}\left(\vec{r}, t'\right). \tag{2.11}$$

In such a linear homogeneous medium follows from eq.(2.1c) for the case of no free charges

$$\int dt' \, \epsilon\left(t-t'\right) \left(\nabla \cdot \vec{E}\left(\vec{r}, t'\right)\right) = 0. \tag{2.12}$$

This is certainly fulfilled for $\vec{\nabla} \cdot \vec{E} = 0$, which simplifies the wave equation (2.4) further

$$\left(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0 \frac{\partial^2}{\partial t^2}\vec{P}. \tag{2.13}$$

This is the wave equation driven by the polarization of the medium. If the medium is linear and has only an induced polarization, completely described in the time domain $\chi\left(t\right)$ or in the frequency domain by its Fourier transform, the complex susceptibility $\tilde{\chi}(\omega) = \tilde{\epsilon}_r(\omega) - 1$ with the relative permittivity $\tilde{\epsilon}_r(\omega) = \tilde{\epsilon}(\omega)/\epsilon_0$, we obtain in the frequency domain with the Fourier transform relationship

$$\widetilde{\vec{E}}(z,\omega) = \int\limits_{-\infty}^{+\infty} \vec{E}(z,t)e^{-j\omega t}dt, \tag{2.14}$$

$$\widetilde{\vec{P}}(\omega) = \epsilon_0 \tilde{\chi}(\omega)\widetilde{\vec{E}}(\omega), \tag{2.15}$$

where, the tildes denote the Fourier transforms in the following. Substituted into (2.13)

$$\left(\Delta + \frac{\omega^2}{c_0^2}\right)\widetilde{\vec{E}}(\omega) = -\omega^2\mu_0\epsilon_0\tilde{\chi}(\omega)\widetilde{\vec{E}}(\omega), \tag{2.16}$$

we obtain

$$\left(\Delta + \frac{\omega^2}{c_0^2}(1 + \tilde{\chi}(\omega))\right)\widetilde{\vec{E}}(\omega) = 0, \tag{2.17}$$

with the refractive index $n(\omega)$ and $1 + \tilde{\chi}(\omega) = n(\omega)^2$ results in the Helmholtz equation

$$\left(\Delta + \frac{\omega^2}{c^2}\right)\widetilde{\vec{E}}(\omega) = 0, \tag{2.18}$$

where $c(\omega) = c_0/n(\omega)$ is the velocity of light in the medium. This equation is the starting point for finding monochromatic wave solutions to Maxwell's equations in linear media, as we will study for different cases in the following. Also, so far we have treated the susceptibility $\tilde{\chi}(\omega)$ as a real quantity, which may not always be the case as we will see later in detail.

2.1.2   Plane-Wave Solutions (TEM-Waves) and Complex Notation

The real wave equation (2.13) for a linear medium has real monochromatic plane wave solutions $\vec{E}_{\vec{k}}(\vec{r}, t)$, which can be be written most efficiently in terms of the complex plane-wave solutions $\underline{\vec{E}}_{\vec{k}}(\vec{r}, t)$ according to

$$\vec{E}_{\vec{k}}(\vec{r}, t) = \frac{1}{2}\left[\underline{\vec{E}}_{\vec{k}}(\vec{r}, t) + \underline{\vec{E}}_{\vec{k}}(\vec{r}, t)^*\right] = \Re e\left\{\underline{\vec{E}}_{\vec{k}}(\vec{r}, t)\right\}, \tag{2.19}$$

with

$$\underline{\vec{E}}_{\vec{k}}(\vec{r}, t) = \underline{E}_{\vec{k}}\ e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\ \vec{e}(\vec{k}). \tag{2.20}$$

Note, we explicitly underlined the complex wave to indicate that this is a complex quantity. Here, $\vec{e}(\vec{k})$ is a unit vector indicating the direction of the electric field which is also called the polarization of the wave, and $\underline{E}_{\vec{k}}$ is the complex field amplitude of the wave with wave vector $\vec{k}$. Substitution of eq.(2.19) into the wave equation results in the dispersion relation, i.e. a

relationship between wave vector and frequency necessary to satisfy the wave equation

$$|\vec{k}|^2 = \frac{\omega^2}{c(\omega)^2} = k(\omega)^2. \tag{2.21}$$

Thus, the dispersion relation is given by

$$k(\omega) = \pm\frac{\omega}{c_0}n(\omega). \tag{2.22}$$

with the wavenumber

$$k = 2\pi/\lambda, \tag{2.23}$$

where λ is the wavelength of the wave in the medium with refractive index n, ω the angular frequency, $\vec{k}$ the wave vector. Note, the natural frequency $f = \omega/2\pi$. From $\nabla \cdot \vec{E} = 0$, for all time, we see that $\vec{k} \perp \vec{e}$. Substitution of the electric field 2.19 into Maxwell's Eqs. (2.1b) results in the magnetic field

$$\vec{H}_{\vec{k}}(\vec{r},t) = \frac{1}{2}\left[\underline{\vec{H}}_{\vec{k}}(\vec{r},t) + \underline{\vec{H}}_{\vec{k}}(\vec{r},t)^*\right] \tag{2.24}$$

with

$$\underline{\vec{H}}_{\vec{k}}(\vec{r},t) = \underline{H}_{\vec{k}}\ e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\ \vec{h}(\vec{k}). \tag{2.25}$$

This complex component of the magnetic field can be determined from the corresponding complex electric field component using Faraday's law

$$-\mathrm{j}\vec{k} \times \left(\underline{E}_{\vec{k}}\ e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\ \vec{e}(\vec{k})\right) = -\mathrm{j}\mu_0\omega\underline{\vec{H}}_{\vec{k}}(\vec{r},t), \tag{2.26}$$

or

$$\underline{\vec{H}}_{\vec{k}}(\vec{r},t) = \frac{\underline{E}_{\vec{k}}}{\mu_0\omega}e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\vec{k} \times \vec{e} = \underline{H}_{\vec{k}}e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\vec{h} \tag{2.27}$$

with

$$\vec{h}(\vec{k}) = \frac{\vec{k}}{|k|} \times \vec{e}(\vec{k}) \tag{2.28}$$

and

$$\underline{H}_{\vec{k}} = \frac{|k|}{\mu_0\omega}\underline{E}_{\vec{k}} = \frac{1}{Z_F}\underline{E}_{\vec{k}}. \tag{2.29}$$

The characteristic impedance of the TEM-wave is the ratio between electric and magnetic field strength

$$Z_F = \mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0\epsilon_r}} = \frac{1}{n}Z_{F_0} \tag{2.30}$$
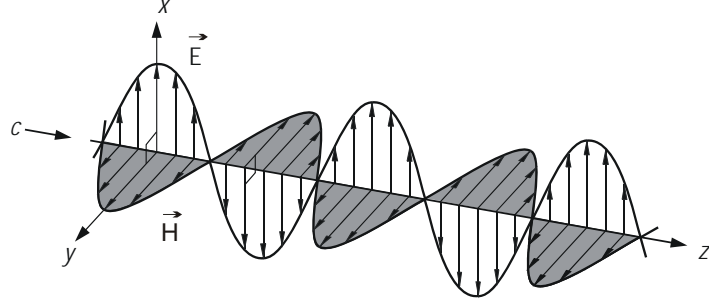
Figure 2.1: Transverse electromagnetic wave (TEM) [6]

with the refractive index $n = \sqrt{\epsilon_r}$ and the free space impedance

$$Z_{F_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\,\Omega. \tag{2.31}$$

Note that the vectors $\vec{e}$, $\vec{h}$ and $\vec{k}$ form an orthogonal trihedral,

$$\vec{e} \perp \vec{h}, \quad \vec{k} \perp \vec{e}, \quad \vec{k} \perp \vec{h}. \tag{2.32}$$

That is why we call these waves transverse electromagnetic (TEM) waves. We consider the electric field of a monochromatic electromagnetic wave with frequency ω and electric field amplitude E_0, which propagates in vacuum along the z-axis, and is polarized along the x-axis, (Fig. 2.1), i.e. $\frac{\vec{k}}{|k|} = \vec{e}_z$, and $\vec{e}(\vec{k}) = \vec{e}_x$. Then we obtain from Eqs.(2.19) and (2.20)

$$\vec{E}(\vec{r}, t) = E_0 \cos(\omega t - kz)\, \vec{e}_x, \tag{2.33}$$

and similiar for the magnetic field

$$\vec{H}(\vec{r}, t) = \frac{E_0}{Z_{F_0}} \cos(\omega t - kz)\, \vec{e}_y, \tag{2.34}$$

see Figure 2.1.Note, that for a backward propagating wave with $\underline{\vec{E}}(\vec{r}, t) = \underline{E}\, e^{\mathrm{j}\omega t + \mathrm{j}\vec{k}\cdot\vec{r}}\, \vec{e}_x$, and $\underline{\vec{H}}(\vec{r}, t) = \underline{H}\, e^{\mathrm{j}(\omega t + \vec{k}\vec{r})}\, \vec{e}_y$, there is a sign change for the magnetic field

$$\underline{H} = -\frac{|k|}{\mu_0 \omega}\underline{E}, \tag{2.35}$$

so that the $(\vec{k}, \vec{E}, \vec{H})$ always form a right handed orthogonal system.

2.1.3 Poynting Vectors, Energy Density and Intensity

The table below summarizes the instantaneous and time averaged energy content and energy transport related to an electromagnetic field

Quantity	Real fields	Complex fields
Electric and magnetic energy density	$w_e = \frac{1}{2}\vec{E} \cdot \vec{D} = \frac{1}{2}\epsilon_0\epsilon_r\vec{E}^2$ $w_m = \frac{1}{2}\vec{H} \cdot \vec{B} = \frac{1}{2}\mu_0\mu_r\vec{H}^2$ $w = w_e + w_m$	$\bar{w}_e = \frac{1}{4}\epsilon_0\epsilon_r\left|\vec{\underline{E}}\right|^2$ $\bar{w}_m = \frac{1}{4}\mu_0\mu_r\left|\vec{\underline{H}}\right|^2$ $\bar{w} = \bar{w}_e + \bar{w}_m$
Poynting vector	$\vec{S} = \vec{E}\times\vec{H}$	$\vec{T} = \frac{1}{2}\vec{\underline{E}}\times\vec{\underline{H}}^*$
Poynting theorem	$\text{div}\vec{S} + \vec{E} \cdot \vec{j} + \frac{\partial w}{\partial t} = 0$	$\text{div}\vec{T} + \frac{1}{2}\vec{\underline{E}} \cdot \vec{\underline{j}}^* +$ $+2j\omega(\bar{w}_m - \bar{w}_e) = 0$
Intensity	$I = \left|\vec{S}\right| = cw$	$I = \text{Re}\{\vec{T}\} = c\bar{w}$

Table 2.1: Poynting vector and energy density in EM-fields

For a plane wave with an electric field $\vec{\underline{E}}(\vec{r}, t) = \underline{E}e^{j(\omega t - kz)}\vec{e}_x$ we obtain for the energy density in units of $[\text{J/m}^3]$

$$w = \frac{1}{2}\epsilon_r\epsilon_0|\underline{E}|^2, \tag{2.36}$$

the complex Poynting vector

$$\vec{T} = \frac{1}{2Z_F}|\underline{E}|^2\,\vec{e}_z, \tag{2.37}$$

and the intensity in units of $[\text{W/m}^2]$

$$I = \frac{1}{2Z_F}|\underline{E}|^2 = \frac{1}{2}Z_F|\underline{H}|^2. \tag{2.38}$$

2.1.4 Classical Permittivity

In this section we want to get insight into propagation of an electromagnetic wavepacket in an isotropic and homogeneous medium, such as a glass optical fiber due to the interaction of radiation with the medium. The electromagnetic properties of a dielectric medium is largely determined by the electric polarization induced by an electric field in the medium. The polarization is
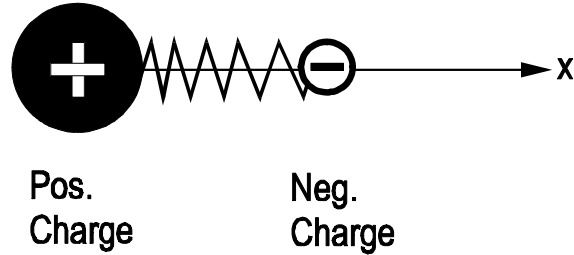
Figure 2.2: Classical harmonic oscillator model for radiation matter interaction

defined as the total induced dipole moment per unit volume. We formulate this directly in the frequency domain

$$\widetilde{\underline{P}}(\omega) = \frac{\text{dipole moment}}{\text{volume}} = N \cdot \langle \widetilde{\underline{p}}(\omega) \rangle = \epsilon_0 \widetilde{\chi}(\omega) \widetilde{\underline{E}}(\omega), \qquad (2.39)$$

where N is density of elementary units and $\langle \underline{\vec{p}} \rangle$ is the average dipole moment of the unit (atom, molecule, ...). In an isotropic and homogeneous medium the induced polarization is proportional to the electric field and the proportionality constant, $\widetilde{\chi}(\omega)$, is called the susceptibility of the medium.

As it turns out (justification later), an electron elastically bound to a positively charged rest atom is not a bad model for understanding the interaction of light with matter at very low electric fields, i.e. the fields do not change the electron distribution in the atom considerably or even ionize the atom, see Figure 2.2. This model is called Lorentz model after the famous physicist A. H. Lorentz (Dutchman) studying electromagnetic phenomena at the turn of the 19th century. He also found the Lorentz Transformation and Invariance of Maxwell's Equations with respect to these transformation, which showed the path to Special Relativity.

The equation of motion for such a unit is the damped harmonic oscillator driven by an electric field in one dimension, x. At optical frequencies, the distance of elongation, x, is much smaller than an optical wavelength (atoms have dimensions on the order of a tenth of a nanometer, whereas optical fields have wavelength on the order of microns) and therefore, we can neglect the spatial variation of the electric field during the motion of the charges within an atom (dipole approximation, i.e. $\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}_A, t) = E(t)\vec{e}_x$).

The equation of motion is

$$m\frac{d^2x}{dt^2} + 2\frac{\Omega_0}{Q}m\frac{dx}{dt} + m\Omega_0^2 x = e_0 E(t), \qquad (2.40)$$

where $\underline{E}(t) = \tilde{E}e^{j\omega t}$. Here, m is the mass of the electron assuming the that the rest atom has infinite mass, e_0 the charge of the electron, Ω_0 is the resonance frequency of the undamped oscillator and Q the quality factor of the resonance, which determines the damping of the oscillator. By using the trial solution $x(t) = \tilde{x}e^{j\omega t}$, we obtain for the complex amplitude of the dipole moment $\tilde{p}$ with the time dependent response $p(t) = e_0 x(t) = \tilde{p}e^{j\omega t}$

$$\tilde{p} = \frac{\frac{e_0^2}{m}}{(\Omega_0^2 - \omega^2) + 2j\frac{\Omega_0}{Q}\omega}\tilde{E}. \qquad (2.41)$$

Note, that we included ad hoc a damping term in the harmonic oscillator equation. At this point it is not clear what the physical origin of this damping term is and we will discuss this at length later in chapter 4. For the moment, we can view this term simply as a consequence of irreversible interactions of the atom with its environment. We then obtain from (2.39) for the susceptibility

$$\underline{\chi}(\omega) = \frac{N\frac{e_0^2}{m}\frac{1}{\epsilon_0}}{(\Omega_0^2 - \omega^2) + 2j\omega\frac{\Omega_0}{Q}} \qquad (2.42)$$

or

$$\widetilde{\chi}(\omega) = \frac{\omega_p^2}{(\Omega_0^2 - \omega^2) + 2j\omega\frac{\Omega_0}{Q}}, \qquad (2.43)$$

with ω_p called the plasma frequency, which is defined as $\omega_p^2 = Ne_0^2/m\epsilon_0$. Figure 2.3 shows the normalized real and imaginary part, $\underline{\widetilde{\chi}}(\omega) = \widetilde{\chi}_r(\omega) + j\widetilde{\chi}_i(\omega)$ of the classical susceptibility (2.43). Note, that there is a small resonance shift (almost invisible) due to the loss. Off resonance, the imaginary part approaches zero very quickly. Not so the real part, which approaches a constant value ω_p^2/Ω_0^2 below resonance for $\omega \to 0$, and approaches zero far above resonance, but much slower than the imaginary part. As we will see later, this is the reason why there are low loss, i.e. transparent, media with refractive index very much different from 1.
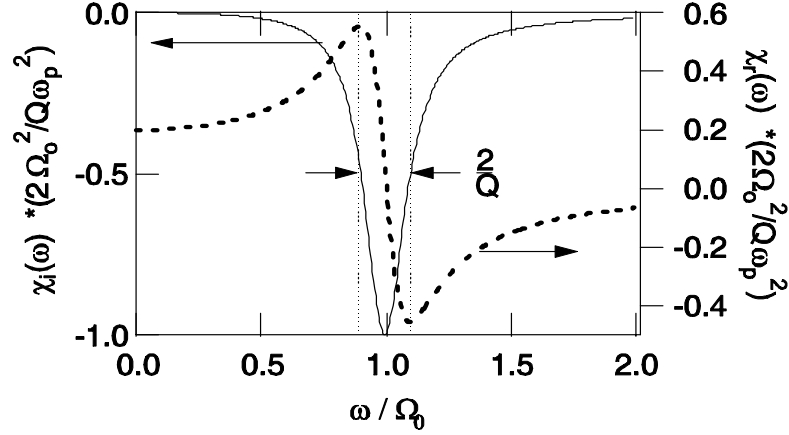
Figure 2.3: Real part (dashed line) and imaginary part (solid line) of the susceptibility of the classical oscillator model for the dielectric polarizability.

2.1.5 Optical Pulses

Optical pulses are wave packets constructed by a continuous superposition of monochromatic plane waves. Consider a TEM-wavepacket, i.e. a superposition of waves with different frequencies, polarized along the x-axis and propagating along the z-axis

$$\vec{\underline{E}}(\vec{r},t) = \int_0^\infty \frac{d\Omega}{2\pi} \widetilde{\underline{E}}(\Omega) e^{j(\Omega t - K(\Omega)z)} \ \vec{e}_x. \tag{2.44}$$

Correspondingly, the magnetic field is given by

$$\vec{\underline{H}}(\vec{r},t) = \int_0^\infty \frac{d\Omega}{2\pi Z_F(\Omega)} \widetilde{\underline{E}}(\Omega) e^{j(\Omega t - K(\Omega)z)} \ \vec{e}_y \tag{2.45}$$

Again, the physical electric and magnetic fields are real and related to the complex fields by

$$\vec{E}(\vec{r},t) = \frac{1}{2} \left(\vec{\underline{E}}(\vec{r},t) + \vec{\underline{E}}(\vec{r},t)^* \right) \tag{2.46}$$

$$\vec{H}(\vec{r},t) = \frac{1}{2} \left(\vec{\underline{H}}(\vec{r},t) + \vec{\underline{H}}(\vec{r},t)^* \right). \tag{2.47}$$

Here, $|\widetilde{\underline{E}}(\Omega)| e^{j\varphi(\Omega)}$ is the complex wave amplitude of the electromagnetic wave at frequency Ω and $K(\Omega) = \Omega/c(\Omega) = n(\Omega)\Omega/c_0$ the wavenumber, where,
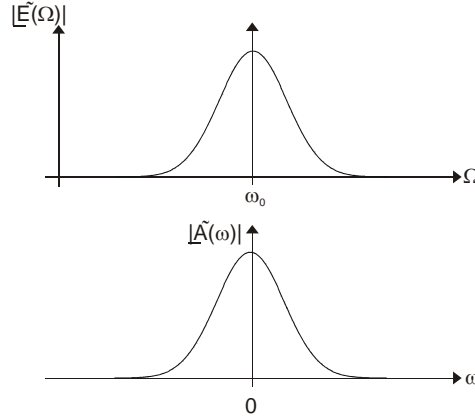
Figure 2.4: Spectrum of an optical wave packet described in absolute and relative frequencies

$n(\Omega)$ is again the refractive index of the medium

$$n^2(\Omega) = 1 + \chi(\Omega), \tag{2.48}$$

c and c_0 are the velocity of light in the medium and in vacuum, respectively. The planes of constant phase propagate with the phase velocity c of the wave.

The wavepacket consists of a superposition of many frequencies with the spectrum shown in Fig. 2.4.

At a given point in space, for simplicity $z = 0$, the complex field of a pulse is given by (Fig. 2.4)

$$\underline{E}(z = 0, t) = \frac{1}{2\pi} \int_0^\infty \underline{\tilde{E}}(\Omega) e^{j\Omega t} d\Omega. \tag{2.49}$$

Optical pulses often have relatively small spectral width compared to the center frequency of the pulse ω_0, as it is illustrated in the upper part of Figure 2.4. For example typical pulses used in optical communication systems for 10Gb/s transmission speed are on the order of 20ps long and have a center wavelength of $\lambda = 1550$nm. Thus the spectral with is only on the order of 50GHz, whereas the center frequency of the pulse is 200THz, i.e. the bandwidth is 4000 smaller than the center frequency. In such cases it is useful to separate the complex electric field in Eq. (2.49) into a carrier frequency ω_0 and an envelope $\underline{A}(t)$ and represent the absolute frequency as

$\Omega = \omega_0 + \omega$. We can then rewrite Eq.(2.49) as

$$
\begin{aligned}
\underline{E}(z &= 0, t) = \frac{1}{2\pi} \int_{-\omega_0}^{\infty} \underline{\tilde{E}}(\omega_0 + \omega) e^{j(\omega_0 + \omega)t} d\omega \qquad (2.50) \\
&= A(t) e^{j\omega_0 t}.
\end{aligned}
$$

The envelope, see Figure 2.8, is given by

$$
\begin{aligned}
\underline{A}(t) &= \frac{1}{2\pi} \int_{-\omega_0 \to -\infty}^{\infty} \underline{\tilde{A}}(\omega) e^{j\omega t} d\omega \qquad (2.51) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{\tilde{A}}(\omega) e^{j\omega t} d\omega, \qquad (2.52)
\end{aligned}
$$

where $\tilde{A}(\omega)$ is the spectrum of the envelope with, $\tilde{A}(\omega) = 0$ for $\omega \leq -\omega_0$. To be physically meaningful, the spectral amplitude $\tilde{A}(\omega)$ must be zero for negative frequencies less than or equal to the carrier frequency, see Figure 2.8. Note, that waves with zero frequency can not propagate, since the corresponding wave vector is zero. The pulse and its envelope are shown in Figure 2.5.
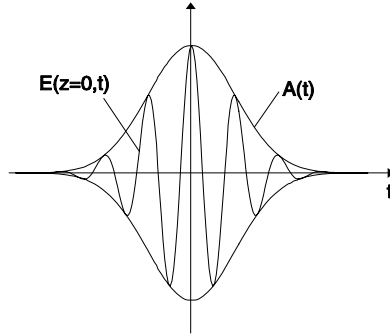

Figure 2.5: Electric field and envelope of an optical pulse.

Table 2.2 shows pulse shape and spectra of some often used pulses as well as the pulse width and time bandwidth products. The pulse width and bandwidth are usually specified as the Full Width at Half Maximum (FWHM) of the intensity in the time domain, $|\underline{A}(t)|^2$, and the spectral density $\left|\underline{\tilde{A}}(\omega)\right|^2$ in the frequency domain, respectively. The pulse shapes and corresponding spectra to the pulses listed in Table 2.2 are shown in Figs 2.6 and 2.7.

Pulse Shape	Fourier Transform	Pulse Width	Time-Bandwidth Product
$\underline{A}(t)$	$\underline{\tilde{A}}(\omega) = \int_{-\infty}^{\infty} a(t)e^{-\mathrm{j}\omega t}dt$	Δt	$\Delta t \cdot \Delta f$
Gaussian: $\ \mathrm{e}^{-\frac{t^2}{2\tau^2}}$	$\sqrt{2\pi}\tau\mathrm{e}^{-\frac{1}{2}\tau^2\omega^2}$	$2\sqrt{\ln 2}\tau$	0.441
Hyperbolic Secant: $\mathrm{sech}(\frac{t}{\tau})$	$\frac{\tau}{2}\ \mathrm{sech}\left(\frac{\pi}{2}\tau\omega\right)$	$1.7627\ \tau$	0.315
Rect-function: $= \begin{cases} 1, & \|t\| \leq \tau/2 \\ 0, & \|t\| > \tau/2 \end{cases}$	$\tau\frac{\sin(\tau\omega/2)}{\tau\omega/2}$	τ	0.886
Lorentzian: $\ \frac{1}{1+(t/\tau)^2}$	$2\pi\tau\mathrm{e}^{-\|\tau\omega\|}$	$1.287\ \tau$	0.142
Double-Exp.: $\ \mathrm{e}^{-\left\|\frac{t}{\tau}\right\|}$	$\frac{\tau}{1+(\omega\tau)^2}$	$\ln 2\ \tau$	0.142

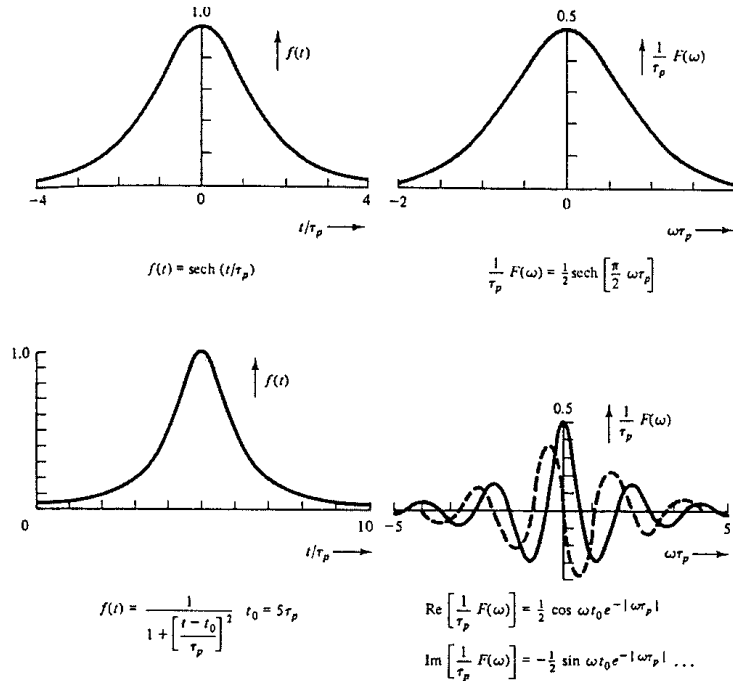Table 2.2: Pulse shapes, corresponding spectra and time bandwidth products.

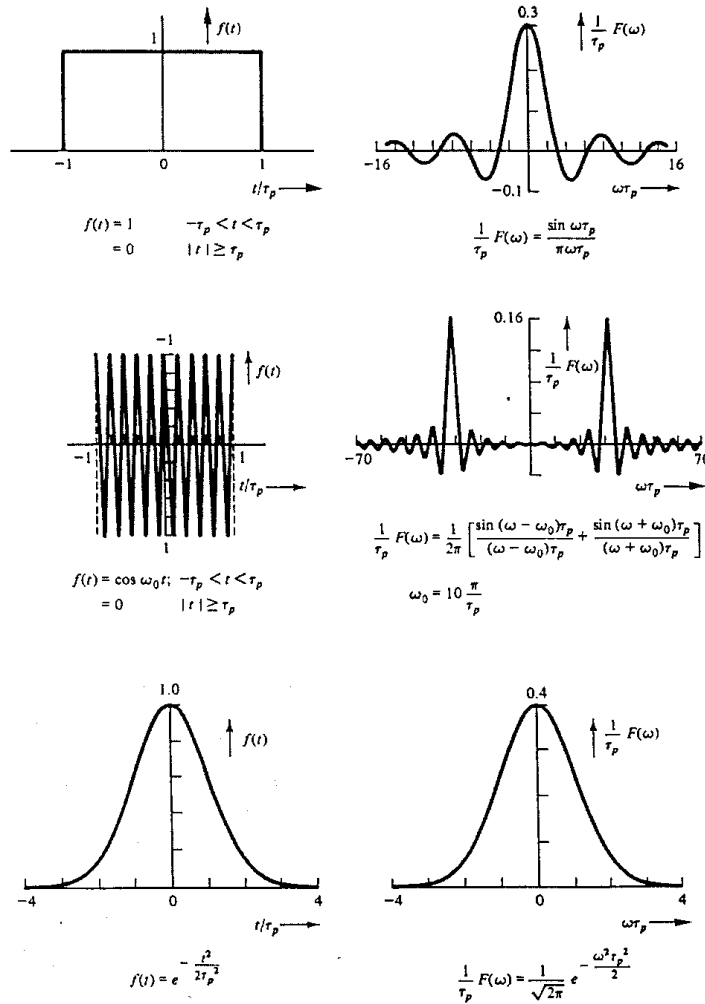Figure 2.6: Fourier transforms to pulse shapes listed in table 2.2 [6].

Figure 2.7: Fourier transforms to pulse shapes listed in table 2.2 continued [6].

2.1.6   Pulse Propagation

Having a basic model for the interaction of light and matter at hand, via section 2.1.4, we can investigate what happens if an electromagnetic wave packet, i.e. an optical pulse propagates through such a medium. We start from Eqs.(2.44) to evaluate the wave packet propagation for an arbitrary

propagation distance z

$$\underline{E}(z,t) = \frac{1}{2\pi} \int_0^\infty \underline{\tilde{E}}(\Omega) e^{\mathrm{j}(\Omega t - K(\Omega)z)} d\Omega. \tag{2.53}$$

Analogous to Eq. (2.50) for a pulse at a given position, we can separate an optical pulse into a carrier wave at frequency ω_0 and a complex envelope $A(z,t)$,

$$\underline{E}(z,t) = \underline{A}(z,t) e^{\mathrm{j}(\omega_0 t - K(\omega_0)z)}. \tag{2.54}$$

By introducing the offset frequency ω, the offset wavenumber $k(\omega)$ and spectrum of the envelope $\underline{\tilde{A}}(\omega)$

$$\omega \;=\; \Omega - \omega_0, \tag{2.55}$$
$$k(\omega) \;=\; K(\omega_0 + \omega) - K(\omega_0), \tag{2.56}$$
$$\underline{\tilde{A}}(\omega) \;=\; \underline{\tilde{E}}(\Omega = \omega_0 + \omega). \tag{2.57}$$

the envelope at propagation distance z, see Fig.2.8, is expressed as

$$\underline{A}(z,t) = \frac{1}{2\pi} \int_{-\infty}^\infty \underline{\tilde{A}}(\omega) e^{\mathrm{j}(\omega t - k(\omega)z)} d\omega, \tag{2.58}$$

with the same constraints on the spectrum of the envelope as before, i.e. the spectrum of the envelope must be zero for negative frequencies beyond the carrier frequency. Depending on the dispersion relation $k(\omega)$, (see Fig. 2.9),.the pulse will be reshaped during propagation as discussed in the following section.

2.1.7 Dispersion

The dispersion relation indicates how much phase shift each frequency component experiences during propagation. These phase shifts, if not linear with respect to frequency, will lead to distortions of the pulse. If the propagation constant $k(\omega)$ is only slowly varying over the pulse spectrum, it is useful to represent the propagation constant, $k(\omega)$, or dispersion relation $K(\Omega)$ by its Taylor expansion, see Fig. 2.9,

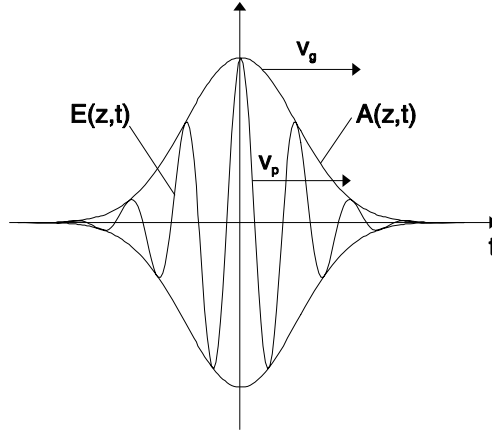$$k(\omega) = k'\omega + \frac{k''}{2}\omega^2 + \frac{k^{(3)}}{6}\omega^3 + O(\omega^4). \tag{2.59}$$

Figure 2.8: Electric field and pulse envelope in time domain.
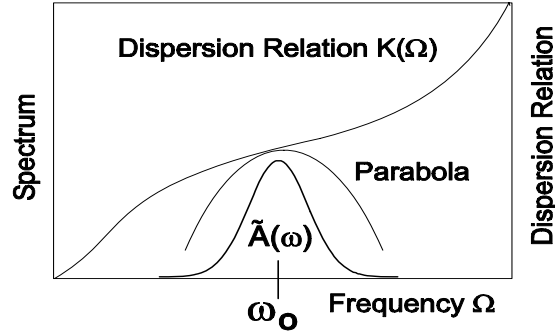

Figure 2.9: Taylor expansion of dispersion relation at the center frequency of the wave packet.

If the refractive index depends on frequency, the dispersion relation is no longer linear with respect to frequency, see Fig. 2.9 and the pulse propagation according to (2.58) can be understood most easily in the frequency domain

$$\frac{\partial \underline{\tilde{A}}(z,\omega)}{\partial z} = -\mathrm{j}k(\omega)\underline{\tilde{A}}(z,\omega). \tag{2.60}$$

Transformation of Eq.() into the time domain gives

$$\frac{\partial \underline{A}(z,t)}{\partial z} = -\mathrm{j}\sum_{n=1}^{\infty}\frac{k^{(n)}}{n!}\left(-\mathrm{j}\frac{\partial}{\partial t}\right)^{n}\underline{A}(z,t). \tag{2.61}$$

If we keep only the first term, the linear term, in Eq.(2.59), then we obtain for the pulse envelope from (2.58) by definition of the group velocity at frequency ω_0

$$v_{g0} = 1/k' = \left(\left. \frac{dk(\omega)}{d\omega} \right|_{\omega=0} \right)^{-1} \tag{2.62}$$

$$\underline{A}(z,t) = \underline{A}(0, t - z/v_{g0}). \tag{2.63}$$

Thus the derivative of the dispersion relation at the carrier frequency determines the propagation velocity of the envelope of the wave packet or group velocity, whereas the ratio between propagation constant and frequency determines the phase velocity of the carrier

$$v_{p0} = \omega_0/K(\omega_0) = \left(\frac{K(\omega_0)}{\omega_0} \right)^{-1}. \tag{2.64}$$

To get rid of the trivial motion of the pulse envelope with the group velocity, we introduce the retarded time $t' = t - z/v_{g0}$. With respect to this retarded time the pulse shape is invariant during propagation, if we approximate the dispersion relation by the slope at the carrier frequency

$$\underline{A}(z,t) = \underline{A}(0, t'). \tag{2.65}$$

Note, if we approximate the dispersion relation by its slope at the carrier frequency, i.e. we retain only the first term in Eq.(2.61), we obtain

$$\frac{\partial \underline{A}(z,t)}{\partial z} + \frac{1}{v_{g0}} \frac{\partial \underline{A}(z,t)}{\partial t} = 0, \tag{2.66}$$

and (2.63) is its solution. If, we transform this equation to the new coordinate system

$$z' = z, \tag{2.67}$$

$$t' = t - z/v_{g0}, \tag{2.68}$$

with

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'} - \frac{1}{v_{g0}} \frac{\partial}{\partial t'}, \tag{2.69}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} \tag{2.70}$$

the transformed equation is

$$\frac{\partial \underline{A}(z',t')}{\partial z'} = 0. \tag{2.71}$$

Since z is equal to z' we keep z in the following.

If the spectrum of the pulse is broad enough, so that the second order term in (2.59) becomes important, the pulse will no longer keep its shape. When keeping in the dispersion relation terms up to second order it follows from (2.58) and (2.69,2.70)

$$\frac{\partial \underline{A}(z,t')}{\partial z} = j\frac{k''}{2}\frac{\partial^2 \underline{A}(z,t')}{\partial t'^2}. \tag{2.72}$$

This is the first non trivial term in the wave equation for the envelope. Because of the superposition principle, the pulse can be thought of to be decomposed into wavepackets (sub-pulses) with different center frequencies. Now, the group velocity depends on the spectral component of the pulse, see Figure 2.10, which will lead to broadening or dispersion of the pulse.
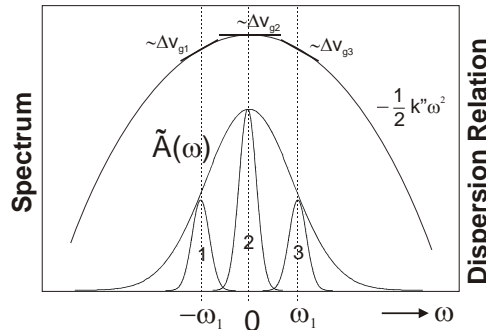

Figure 2.10: Decomposition of a pulse into wave packets with different center frequency. In a medium with dispersion the wavepackets move at different relative group velocity.

Fortunately, for a Gaussian pulse, the pulse propagation equation 2.72 can be solved analytically. The initial pulse is then of the form

$$\underline{E}(z = 0,t) = \underline{A}(z=0,t)e^{j\omega_0 t} \tag{2.73}$$

$$\underline{A}(z = 0, t = t') = \underline{A}_0 \exp\left[-\frac{1}{2}\frac{t'^2}{\tau^2}\right] \tag{2.74}$$

Eq.(2.72) is most easily solved in the frequency domain since it transforms to

$$\frac{\partial \tilde{\underline{A}}(z,\omega)}{\partial z} = -j\frac{k''\omega^2}{2}\tilde{\underline{A}}(z,\omega), \tag{2.75}$$

with the solution

$$\tilde{\underline{A}}(z,\omega) = \tilde{\underline{A}}(z=0,\omega)\exp\left[-j\frac{k''\omega^2}{2}z\right]. \tag{2.76}$$

The pulse spectrum aquires a parabolic phase. Note, that here ω is the Fourier Transform variable conjugate to t' rather than t. The Gaussian pulse has the advantage that its Fourier transform is also a Gaussian

$$\tilde{\underline{A}}(z=0,\omega) = A_0\sqrt{2\pi}\tau\exp\left[-\frac{1}{2}\tau^2\omega^2\right] \tag{2.77}$$

and, therefore, in the spectral domain the solution at an arbitray propagation distance z is

$$\tilde{\underline{A}}(z,\omega) = A_0\sqrt{2\pi}\tau\exp\left[-\frac{1}{2}\left(\tau^2+jk''z\right)\omega^2\right]. \tag{2.78}$$

The inverse Fourier transform is analogously

$$\underline{A}(z,t') = A_0\left(\frac{\tau^2}{(\tau^2+jk''z)}\right)^{1/2}\exp\left[-\frac{1}{2}\frac{t'^2}{(\tau^2+jk''z)}\right] \tag{2.79}$$

The exponent can be written as real and imaginary part and we finally obtain

$$\underline{A}(z,t') = A_0\left(\frac{\tau^2}{(\tau^2+jk''z)}\right)^{1/2}\exp\left[-\frac{1}{2}\frac{\tau^2 t'^2}{\left(\tau^4+(k''z)^2\right)}+j\frac{1}{2}k''z\frac{t'^2}{\left(\tau^4+(k''z)^2\right)}\right] \tag{2.80}$$

As we see from Eq.(2.80) during propagation the FWHM of the Gaussian determined by

$$\exp\left[-\frac{\tau(\tau'_{FWHM}/2)^2}{\left(\tau^4+(k''z)^2\right)}\right] = 0.5 \tag{2.81}$$

changes from

$$\tau_{FWHM} = 2\sqrt{\ln 2}\ \tau \tag{2.82}$$

at the start to

$$\tau'_{FWHM} \;=\; 2\sqrt{\ln 2}\,\tau\sqrt{1+\left(\frac{k''L}{\tau^2}\right)^2} \tag{2.83}$$

$$=\; \tau_{FWHM}\sqrt{1+\left(\frac{k''L}{\tau^2}\right)^2}$$

at $z = L$. For large stretching this result simplifies to

$$\tau'_{FWHM} = 2\sqrt{\ln 2}\,\left|\frac{k''L}{\tau}\right| \ \text{ for } \ \left|\frac{k''L}{\tau^2}\right| \gg 1. \tag{2.84}$$

The strongly dispersed pulse has a width equal to the difference in group delay over the spectral width of the pulse.

Figure 2.11 shows the evolution of the magnitude of the Gaussian wave packet during propagation in a medium which has no higher order dispersion in normalized units. The pulse spreads continuously.
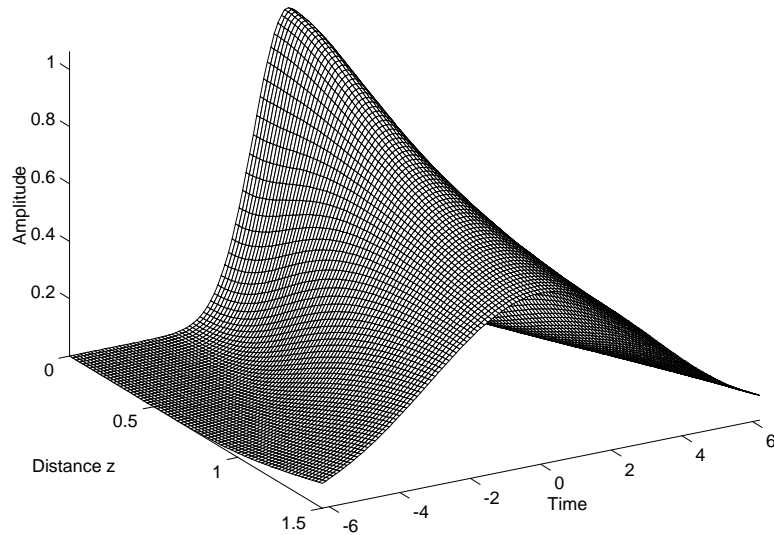

Figure 2.11: Magnitude of the complex envelope of a Gaussian   pulse, $|\underline{A}(z, t')|$, in a dispersive medium.

As discussed before, the origin of this spreading is the group velocity dispersion (GVD), $k'' \neq 0$. The group velocity varies over the pulse spectrum significantly leading to a group delay dispersion (GDD) after a propagation distance $z = L$ of $k''L \neq 0$, for the different frequency components. This leads to the build-up of chirp in the pulse during propagation. We can understand this chirp by looking at the parabolic phase that develops over the pulse in time at a fixed propagation distance. The phase is, see Eq.(2.80)

$$\phi(z = L, t') = -\frac{1}{2} \arctan \left[\frac{k''L}{\tau^2} \right] + \frac{1}{2} k''L \frac{t'^2}{\left(\tau^4 + (k''L)^2 \right)}. \tag{2.85}$$
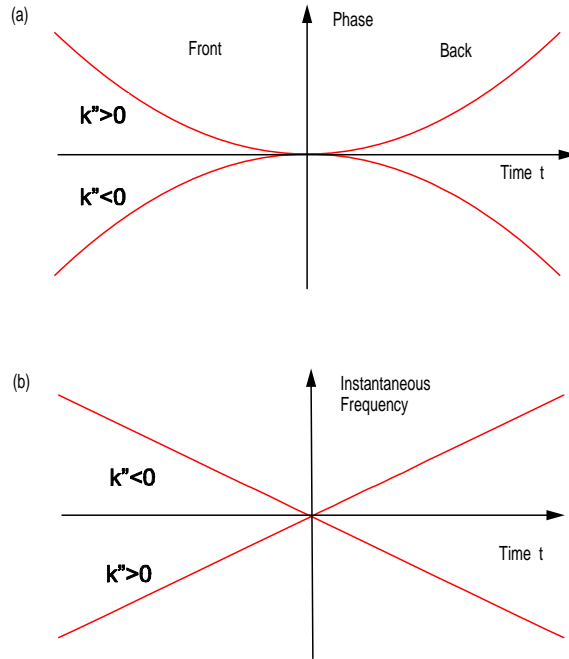

Figure 2.12: (a) Phase and (b) instantaneous frequency of a Gaussian pulse during propagation through a medium with positive or negative dispersion.

This parabolic phase, see Fig. 2.12 (a), can be understood as a localy varying frequency in the pulse, i.e. the derivative of the phase gives the

instantaneous frequency shift in the pulse with respect to the center frequency

$$\omega(z = L, t') = \frac{\partial}{\partial t'} \phi(L, t') = \frac{k''L}{\left(\tau^4 + (k''L)^2\right)} t' \tag{2.86}$$

see Fig.2.12 (b). The instantaneous frequency indicates that for a medium with positive GVD, ie. $k'' > 0$, the low frequencies are in the front of the pulse, whereas the high frequencies are in the back of the pulse, since the sub-pulses with lower frequencies travel faster than sub-pulses with higher frequencies. The opposite is the case for negative dispersive materials.

It is instructive for later purposes, that this behaviour can be completely understood from the center of mass motion of the sub-pulses, see Figure 2.10. Note, we can choose a set of sub-pulses, with such narrow bandwidth, that dispersion does not matter. In the time domain, these pulses are of course very long, because of the time bandwidth relationship. Nevertheless, since they all have different carrier frequencies, they interfere with each other in such a way that the superposition is a very narrow pulse. This interference, becomes destroyed during propagation, since the sub-pulses propagate at different speed, i.e. their center of mass propagates at different speed.

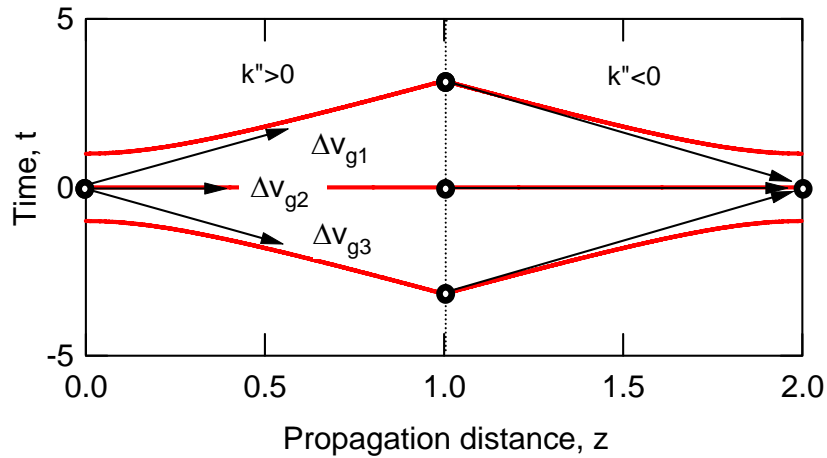Figure 2.13: Pulse spreading by following the center of mass of sub-pulses according to Fig. 2.10. For $z < 1$, the pulses propagate in a medium with positive dispersion and for $z > 1$ in a medium with negative dispersion.

The differential group delay $\Delta T_g(\omega) = k''L\omega$ of a sub-pulse with its center frequency ω different from 0, is due to its differential group velocity $\Delta v_g(\omega) = -v_{g0}\Delta T_g(\omega)/T_{g0} = -v_{g0}^2 k''\omega$. Note, that $T_{g0} = L/v_{g0}$. This is illustrated in Figure 2.13 by ploting the trajectory of the relative motion of the center of mass of each sub-pulse as a function of propagation distance, which asymptotically approaches the formula for the pulse width of the highly dispersed pulse Eq.(2.84). If we assume that the pulse propagates through a negative dispersive medium following the positive dispersive medium, the group velocity of each sub-pulse is reversed. The sub-pulses propagate towards each other until they all meet at one point (focus) to produce again a short and unchirped initial pulse, see Figure 2.13. This is a very powerful technique to understand dispersive wave motion and as we will see in the next section is the connection between ray optics and physical optics.

2.1.8 Loss and Gain

If the medium considered has loss, described by the imaginary part of the dielectric susceptibility, see (2.43) and Fig. 2.3, we can incorporate this loss into a complex refractive index

$$\underline{n}(\Omega) = n_r(\Omega) + jn_i(\Omega) \tag{2.87}$$

via

$$\underline{n}(\Omega) = \sqrt{1 + \underline{\widetilde{\chi}}(\Omega)}. \tag{2.88}$$

For an optically thin medium, i.e. $\underline{\widetilde{\chi}} \ll 1$ the following approximation is very useful

$$\underline{n}(\Omega) \approx 1 + \frac{\underline{\widetilde{\chi}}(\Omega)}{2}. \tag{2.89}$$

As one can show (in Recitations) the complex susceptibility (2.43) can be approximated close to resonance, i.e. $\Omega \approx \Omega_0$, by the complex Lorentzian lineshape

$$\underline{\widetilde{\chi}}(\Omega) = \frac{-j\chi_0}{1 + jQ\frac{\Omega - \Omega_0}{\Omega_0}}, \tag{2.90}$$

where $\chi_0 = Q\frac{\omega_p^2}{2\Omega_0^2}$ will turn out to be related to the peak absorption of the line, which is proportional to the density of atoms, Ω_0 is the center frequency

and $\Delta\Omega = \frac{\Omega_0}{Q}$ is the half width half maximum (HWHM) linewidth of the transition. The real and imaginary part of the complex Lorentzian are

$$\widetilde{\chi}_r(\Omega) \quad = \quad \frac{-\chi_0 \frac{(\Omega - \Omega_0)}{\Delta\Omega}}{1 + \left(\frac{\Omega - \Omega_0}{\Delta\Omega}\right)^2}, \tag{2.91}$$

$$\widetilde{\chi}_i(\Omega) \quad = \quad \frac{-\chi_0}{1 + \left(\frac{\Omega - \Omega_0}{\Delta\Omega}\right)^2}, . \tag{2.92}$$

In the derivation of the wave equation for the pulse envelope (2.61) in section 2.1.7, there was no restriction to a real refractive index. Therefore, the wave equation (2.61) also treats the case of a complex refractive index. If we assume a medium with the complex refractive index (2.89), then the wavenumber is given by

$$\underline{K}(\Omega) = \frac{\Omega}{c_0}\left(1 + \frac{1}{2}\left(\widetilde{\chi}_r(\Omega) + \mathrm{j}\widetilde{\chi}_i(\Omega)\right)\right). \tag{2.93}$$

Since we introduced a complex wavenumber, we have to redefine the group velocity as the inverse derivative of the real part of the wavenumber with respect to frequency. At line center, we obtain

$$v_g^{-1} = \left.\frac{\partial K_r(\Omega)}{\partial \Omega}\right|_{\Omega_0} = \frac{1}{c_0}\left(1 - \frac{\chi_0}{2}\frac{\Omega_0}{\Delta\Omega}\right). \tag{2.94}$$

Thus, for a narrow absorption line, $\chi_0 > 0$ and $\frac{\Omega_0}{\Delta\Omega} \gg 1$, the absolute value of the group velocity can become much larger than the velocity of light in vacuum. The opposite is true for an amplifying medium, $\chi_0 < 0$. There is nothing wrong with this finding, since the group velocity only describes the motion of the peak of a Gaussian wave packet, which is not a causal wave packet. A causal wave packet is identical to zero for some earlier time $t < t_0$, in some region of space. A Gaussian wave packet fills the whole space at any time and can be reconstructed by a Taylor expansion at any time. Therefore, the tachionic motion of the peak of such a signal does not contradict special relativity.

The imaginary part in the wave vector (2.93) leads with $K = \frac{\Omega}{c_0}$ to absorption

$$\alpha(\Omega) = -K\widetilde{\chi}_i(\Omega). \tag{2.95}$$

In the envelope equation (2.60) for a wavepacket with carrier frequency $\omega_0 = \Omega_0$ and $K_0 = \frac{\Omega_0}{c_0}$ the loss leads to a term of the form

$$\left.\frac{\partial \tilde{\underline{A}}(z,\omega)}{\partial z}\right|_{(loss)} = -\alpha(\Omega_0 + \omega)\tilde{A}(z,\omega) = \frac{-\chi_0 K_0}{1 + \left(\frac{\omega}{\Delta\Omega}\right)^2}\tilde{\underline{A}}(z,\omega). \qquad (2.96)$$

In the time domain, we obtain up to second order in the inverse linewidth

$$\left.\frac{\partial \underline{A}(z,t')}{\partial z}\right|_{(loss)} = -\chi_0 K_0 \left(1 + \frac{1}{\Delta\Omega^2}\frac{\partial^2}{\partial t^2}\right)\underline{A}(z,t'), \qquad (2.97)$$

which corresponds to a parabolic approximation of the line shape at line center, (Fig. 2.3). As we will see later, for an amplifying optical transition we obtain a similar equation. We only have to replace the loss by gain

$$\left.\frac{\partial \underline{A}(z,t')}{\partial z}\right|_{(gain)} = g\left(1 + \frac{1}{\Omega_g^2}\frac{\partial^2}{\partial t^2}\right)\underline{A}(z,t'), \qquad (2.98)$$

where $g = -\chi_0 K_0$ is the peak gain at line center per unit length and Ω_g is the HWHM linewidth of a transition providing gain.

2.1.9   Sellmeier Equation and Kramers-Kroenig Relations

The linear susceptibility is the frequency response or impulse response of a linear system to an applied electric field, see Eq.(2.41). For a real physical system this response is causal, and therefore real and imaginary parts obey Kramers-Kroenig Relations

$$\chi_r(\Omega) = \frac{2}{\pi}\int_0^\infty \frac{\omega\chi_i(\omega)}{\omega^2 - \Omega^2}d\omega = n_r^2(\Omega) - 1, \qquad (2.99)$$

$$\chi_i(\Omega) = -\frac{2}{\pi}\int_0^\infty \frac{\Omega\chi_r(\omega)}{\omega^2 - \Omega^2}d\omega. \qquad (2.100)$$

For optical media these relations have the consequence that the refractive index and absorption of a medium are not independent, which can often be exploited to compute the index from absorption data or the other way

around. The Kramers-Kroenig Relations also give us a good understanding of the index variations in transparent media, which means the media are used in a frequency range far away from resonances. Then the imaginary part of the susceptibility related to absorption can be approximated by

$$\chi_i(\Omega) = \sum_i A_i \delta \left(\omega - \omega_i \right) \tag{2.101}$$

and the Kramers-Kroenig relation results in the Sellmeier Equation for the refractive index

$$n^2(\Omega) = 1 + \sum_i A_i \frac{\omega_i}{\omega_i^2 - \Omega^2} = 1 + \sum_i a_i \frac{\lambda}{\lambda^2 - \lambda_i^2}. \tag{2.102}$$

This formula is very useful in fitting the refractive index of various media over a large frequency range with relatively few coefficients. For example Table 2.3 shows the sellmeier coefficients for fused quartz and sapphire.

	Fused Quartz	Sapphire
a_1	0.6961663	1.023798
a_2	0.4079426	1.058364
a_3	0.8974794	5.280792
λ_1^2	$4.679148 \cdot 10^{-3}$	$3.77588 \cdot 10^{-3}$
λ_2^2	$1.3512063 \cdot 10^{-2}$	$1.22544 \cdot 10^{-2}$
λ_3^2	$0.9793400 \cdot 10^{2}$	$3.213616 \cdot 10^{2}$

Table 2.3: Table with Sellmeier coefficients for fused quartz and sapphire.

In general, each absorption line contributes a corresponding index change to the overall optical characteristics of a material, see Fig. 2.14. A typical situation for a material having resonances in the UV and IR, such as glass, is shown in Fig. 2.15. As Fig. 2.15 shows, due to the Lorentzian line shape, that outside of an absorption line the refractive index is always decreasing as a function of wavelength. This behavior is called normal dispersion and the opposite behavior abnormal dispersion.

$$\frac{dn}{d\lambda} \quad < \quad 0 : \text{normal dispersion (blue refracts more than red)}$$

$$\frac{dn}{d\lambda} \quad > \quad 0 : \text{abnormal dispersion}$$

This behavior is also responsible for the mostly positive group delay dispersion over the transparency range of a material, as the group velocity or group delay dispersion is closely related to $\frac{dn}{d\lambda}$. Fig.2.16 shows the transparency range of some often used media.
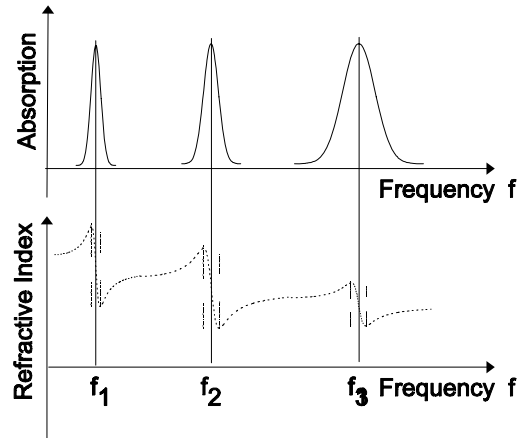
Figure 2.14: Each absorption line must contribute to an index change via the Kramers-Kroenig relations.
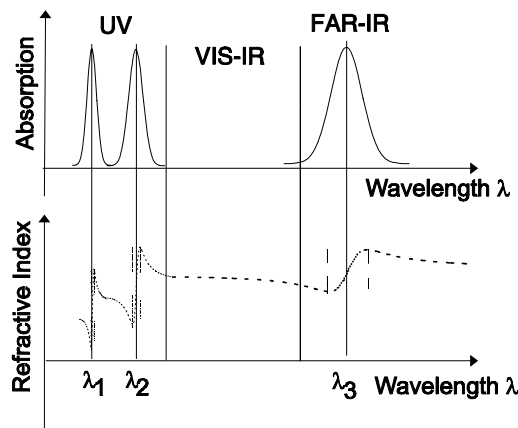
Figure 2.15: Typcial distribution of absorption lines in a medium transparent in the visible.
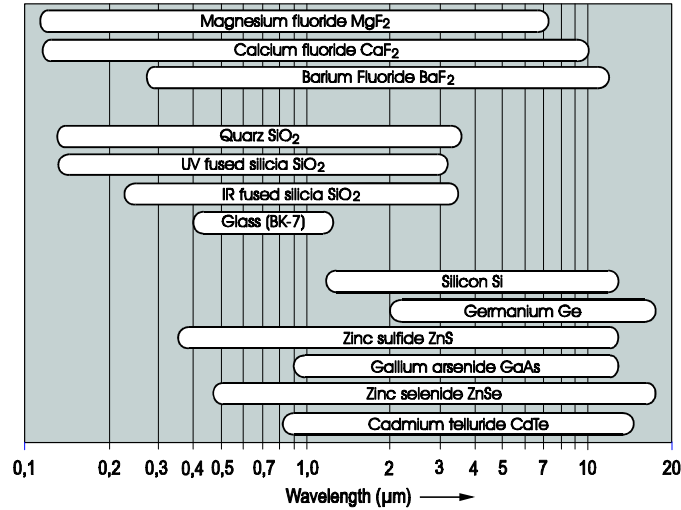
Figure 2.16: Transparency range of some materials according to [6], p. 175.

Often the dispersion GVD and GDD needs to be calculated from the Sellmeier equation, i.e. $n(\lambda)$. The corresponding quantities are listed in Table 2.4. The computations are done by substituting the frequency with the wavelength.

Dispersion Characteristic	Definition	Comp. from $n(\lambda)$
medium wavelength: λ_n	$\frac{\lambda}{n}$	$\frac{\lambda}{n(\lambda)}$
wavenumber: k	$\frac{2\pi}{\lambda_n}$	$\frac{2\pi}{\lambda}n(\lambda)$
phase velocity: v_p	$\frac{\omega}{k}$	$\frac{c_0}{n(\lambda)}$
group velocity: v_g	$\frac{d\omega}{dk}; d\lambda = \frac{-\lambda^2}{2\pi c_0}d\omega$	$\frac{c_0}{n}\left(1 - \frac{\lambda}{n}\frac{dn}{d\lambda}\right)^{-1}$
group velocity dispersion: GVD	$\frac{d^2k}{d\omega^2}$	$\frac{\lambda^3}{2\pi c_0^2}\frac{d^2n}{d\lambda^2}$
group delay: $T_g = \frac{L}{v_g} = \frac{d\phi}{d\omega}$	$\frac{d\phi}{d\omega} = \frac{d(kL)}{d\omega}$	$\frac{n}{c_0}\left(1 - \frac{\lambda}{n}\frac{dn}{d\lambda}\right)L$
group delay dispersion: GDD	$\frac{dT_g}{d\omega} = \frac{d^2(kL)}{d\omega^2}$	$\frac{\lambda^3}{2\pi c_0^2}\frac{d^2n}{d\lambda^2}L$

Table 2.4: Table with important dispersion characteristics and how to compute them from the wavelength dependent refractive index $n(\lambda)$.