2.8 Wave Propagation in Anisotropic Media

So far we have always assumed that the medium in which the electromagnetic wave propagates is isotropic. This causes the induced polarization to be parallel to the applied electric field. In crystaline materials or materials with microscopic fine structure in general, this is no longer the case. Instead of the simple relation

$$\underline{\vec{P}} = \epsilon_0 \underline{\chi} \cdot \underline{\vec{E}}, \tag{2.390}$$

where the susceptibility is a scalar, the induced polarization may have a general lineare dependence on $\vec{E}$ not necessarily parallel to the applied field

$$\underline{P}_x = \epsilon_0 \left(\underline{\chi}_{xx} \underline{E}_x + \underline{\chi}_{xy} \underline{E}_y + \underline{\chi}_{xz} \underline{E}_z \right), \tag{2.391}$$

$$\underline{P}_y = \epsilon_0 \left(\underline{\chi}_{yx} \underline{E}_x + \underline{\chi}_{yy} \underline{E}_y + \underline{\chi}_{yz} \underline{E}_z \right), \tag{2.392}$$

$$\underline{P}_z = \epsilon_0 \left(\underline{\chi}_{zx} \underline{E}_x + \underline{\chi}_{zy} \underline{E}_y + \underline{\chi}_{zz} \underline{E}_z \right). \tag{2.393}$$

The tensor $\underline{\chi}$ is called the electric susceptibility tensor. As shown in Table 2.7 the crytaline structure determines to a large extend the values of the susceptibility tensor elements or in other words the symmetry properties of

isotropic	$\begin{bmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & xx \end{bmatrix}$	cubic
uniaxial	$\begin{bmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{bmatrix}$	Tetragonal Trigonal Hexagonal
biaxial	$\begin{bmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{bmatrix}$	Orthorhombic
	$\begin{bmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ xz & 0 & zz \end{bmatrix}$	Monoclinic
	$\begin{bmatrix} xx & xy & xz \\ xy & yy & yz \\ xz & yz & zz \end{bmatrix}$	Triclinic

Table 2.7: Form of the electric susceptibility tensor for various crystal systems.

the crystal reflect themselves in the symmetry properties of the susceptibility tensor.

Elementary algebra tells us that we can choose a new coordinate system with axis x', y', z', such that the susceptibility tensor has diagonal form

$$\underline{P}_{x'} = \epsilon_0 \underline{\chi}_{x'x'} \underline{E}_{x'}, \quad (2.394)$$

$$\underline{P}_{y'} = \epsilon_0 \underline{\chi}_{y'y'} \underline{E}_{y'}, \quad (2.395)$$

$$\underline{P}_{z'} = \epsilon_0 \underline{\chi}_{z'z'} \underline{E}_{z'}. \quad (2.396)$$

These directions are called the principle axes of the crystal. In the following, we consider that the crystal axes are aligned with the principle axes. If a TEM-wave is launched along the $z-$axis with the elecric field polarized along one of the prinicple axes, lets say x, the wave will experience a refractive index

$$n_x^2 = 1 + \underline{\chi}_{xx} \quad (2.397)$$

and the wave will have a phase velocity

$$c = c_0/n_x. \quad (2.398)$$

If on the other hand the wave is polarized along the y-axis it will have a different phase velocity corresponding to n_y. If the wave propagates along the $z-$axis with electric field components along both the x- and y-axis, the wave can be decomposed into the two polarization components. During propagation of the wave the will experience a differential phase shift with respect to each other and the state of polarization may change. Later, this phenomenon will be exploited for the construction of modulators and switches.

2.8.1 Birefringence and Index Ellipsoid

If we consider the propagation of a wave into an arbitrary direction of the crystal it is no longer obvious what the plane wave solution and its phase velocity is. We have

$$\vec{D} = \boldsymbol{\varepsilon}\vec{E} \tag{2.399}$$

with

$$\boldsymbol{\varepsilon} = \varepsilon_0 \begin{bmatrix} \boldsymbol{\varepsilon}_x & 0 & 0 \\ 0 & \boldsymbol{\varepsilon}_y & 0 \\ 0 & 0 & \boldsymbol{\varepsilon}_z \end{bmatrix}. \tag{2.400}$$

Let's assume there are plane wave solutions

$$\underline{\vec{E}} = \underline{\vec{E}}_0 \mathrm{e}^{-j\vec{k}\cdot\vec{r}}$$

then Ampere's and Faraday's law give

$$\vec{k} \times \underline{\vec{H}} = -\omega\boldsymbol{\varepsilon}\underline{\vec{E}}, \tag{2.401}$$

$$\vec{k} \times \underline{\vec{E}} = \omega\mu_0\underline{\vec{H}}, \tag{2.402}$$

resulting in the wave equation

$$\vec{k} \times \vec{k} \times \underline{\vec{E}} = -\omega^2\mu_0\boldsymbol{\varepsilon}\underline{\vec{E}}. \tag{2.403}$$

Note, that the wavevector $\vec{k}$ is orthogonal to the dielectric displacement $\vec{D}$ and the magnetic field $\vec{H}$, but not necessarily to the electric field $\vec{E}$. There is

$$\vec{k}\perp\left(\boldsymbol{\varepsilon}\vec{E} = \vec{D}\right)\perp\vec{B}. \tag{2.404}$$

This situation is reflected in Figure 2.114

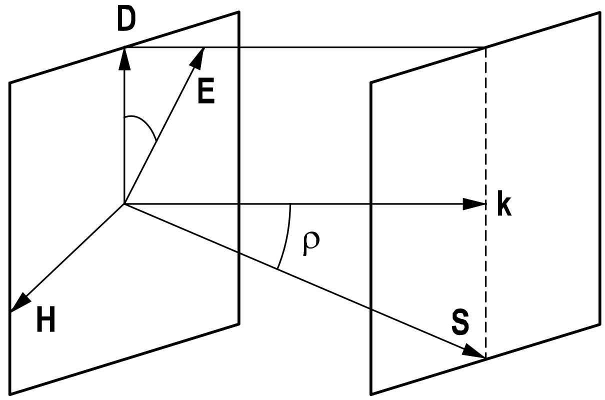

Figure 2.114: Wave propagation in anisotropic media. KDB-system.

One distinguishes between isotropic, uniaxial und biaxial media. We have extensively studied the isotropic case. The most general case is the biaxial case, where the dielectric constants along the three axes are all different. These dielectic constants, or corresponding indices, define an index ellipsoid

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1, \tag{2.405}$$

see Figure 2.115.

Here we want to consider the case of an uniaxial crystal, where

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_1 \neq \varepsilon_{zz} = \varepsilon_3. \tag{2.406}$$

The refractive indices corresponding to these susceptibilities are called ordinary and extraordinary indices

$$n_1 = n_o \neq n_3 = n_e. \tag{2.407}$$

Further, there is a distinction between positive, $n_e > n_o$, and negative, $n_e < n_o$, uniaxial crystals. The uniaxial case corresponds to an index ellipsoid that has rotational symmectry around the z-axis, see Figure 2.115.

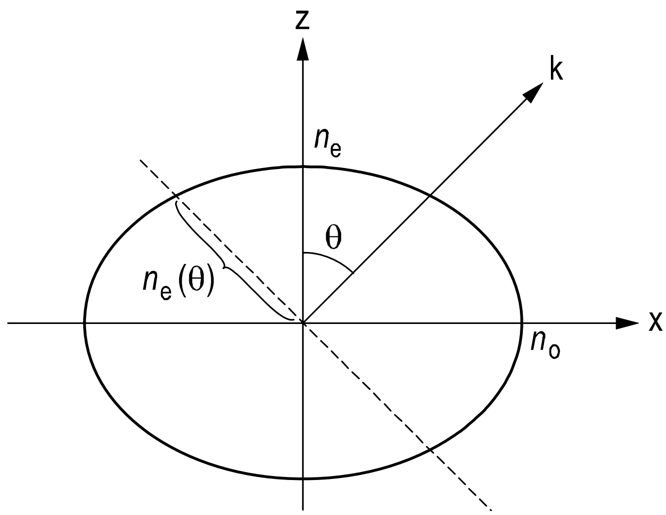

Figure 2.115: Index Ellipsoid

The general case is then a wave with wave vector $\vec{k}$ propagating under an angle θ with respect to the z-axis; the z-axis is also often called the fast axis or c-axis or optical axis. Without restrictions, we assume that the wave vector is in the $x-z-$plane. If the wave vector is aligned with the fast axis, there is no birefringence, because the index experienced by the wave is independent from its polarization. If there is a finite angle, $\theta \neq 0$, then there are two waves with different phase velocity and group velocity as we will show now, see 2.115, and birefringence occurs. With the identity $\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \left(\vec{A} \cdot \vec{C}\right)\vec{B} - \left(\vec{A} \cdot \vec{B}\right)\vec{C}$, when applied to Eq.(2.403), follows

$$\left(\vec{k} \cdot \vec{E}\right)\vec{k} - k^2\vec{E} + \omega^2\mu_0\boldsymbol{\varepsilon}\vec{E} = 0. \tag{2.408}$$

This equation determines the dispersion relation and polarization of the possible waves with wave vector $\vec{k}$. Since the wave vector is in the $x-z-$plane this equation reads

$$\begin{pmatrix} k_0^2n_o^2 + k_x^2 - k^2 & & k_xk_z \\ & k_0^2n_o^2 - k^2 & \\ k_zk_x & & k_0^2n_e^2 + k_z^2 - k^2 \end{pmatrix}\vec{E} = 0 \tag{2.409}$$

This equation clearly shows that a wave polarized along the $y-$axis or in general orthogonal to the plane composed of the wave vector and the fast axis decouples from the other components.

2.8.2 Ordinary Wave

This wave is called the ordinary wave, because it has the dispersion relation

$$k^2 = k_0^2 n_o^2. \tag{2.410}$$

As with the TEM waves in an isotropic medium, the wave vector and the field components build an orthogonal trihedral, $\vec{k} \perp \vec{E} \perp \vec{H}$.

2.8.3 Extraordinary Wave

Eq.(2.409) allows for another wave with a polarization in the $x-z-$plane, and therefore this wave has a longitudinal electric field component. This wave is called extraordinary wave and its dispersion relation follows from

$$\det \begin{vmatrix} k_0^2 n_o^2 + k_x^2 - k^2 & k_x k_z \\ k_z k_x & k_0^2 n_e^2 + k_z^2 - k^2 \end{vmatrix} = \mathbf{0}. \tag{2.411}$$

Calculating the determinant and simplifying we find

$$\frac{k_z^2}{n_o^2} + \frac{k_x^2}{n_e^2} = k_0^2. \tag{2.412}$$

With $k_x = k \sin(\theta)$, $k_z = k \cos(\theta)$ and $k = n(\theta) k_0$ we obtain for the refractive index seen by the extraordinary wave

$$\frac{1}{n(\theta)^2} = \frac{\cos^2(\theta)}{n_o^2} + \frac{\sin^2(\theta)}{n_e^2}. \tag{2.413}$$

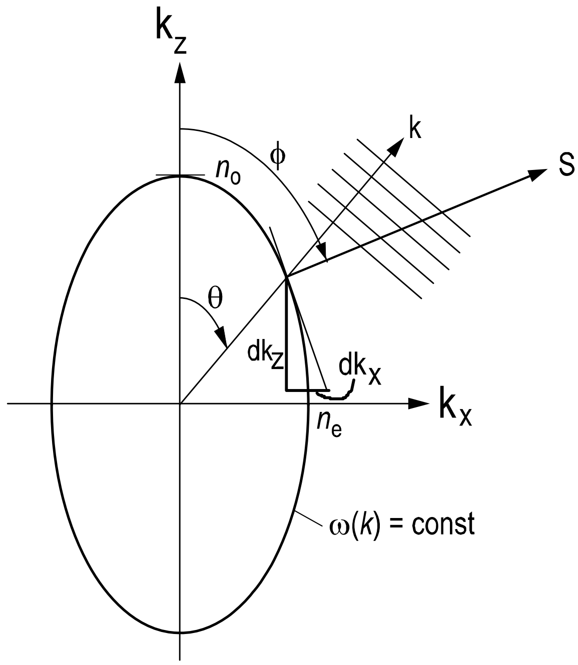

Figure 2.116: Cut through the surface with a constant free space wave number $k_o(k_x, k_y, k_z)$ or frequency, which is also an ellipsoid, but with exchanged principle axis when compared with Figure 2.114

Eqs.(2.412) and (2.413) also describe an ellipse. This ellipse is the location of a constant free space wave number or frequency, $\omega = k_0 c_0$, and therefore determines the refractive index, $n(\theta)$, of the extraordinary wave, see Figure 2.115. The group velocity is found to be parallel to the Poynting vector

$$\boldsymbol{v}_g = \nabla_k \omega(\mathbf{k}) \parallel \mathbf{S}, \tag{2.414}$$

and is orthogonal to the surface. For completeness, we give a derivation of the walk-off angle between the ordinary and extraordinary wave

$$\tan\theta = \frac{k_x}{k_z} \tag{2.415}$$

$$\tan\phi = -\frac{dk_z}{dk_x} \tag{2.416}$$

From Eq.(2.412) we obtain by differentiation along the surface of the ellipsoid

$$\frac{2k_z dk_z}{n_o^2} + \frac{2k_x dk_x}{n_e^2} = 0. \tag{2.417}$$

$$\tan\phi = \frac{n_o^2 k_x}{n_e^2 k_z} = \frac{n_o^2}{n_e^2}\tan\theta$$

Thus, we obtain for the walk-off-angle ϱ between Poynting vector and wave vector

$$\tan\varrho = \tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} \tag{2.418}$$

or

$$\tan\varrho = -\frac{\left(\frac{n_0^2}{n_e^2} - 1\right)\tan\theta}{1 + \frac{n_0^2}{n_e^2}\tan^2\theta}. \tag{2.419}$$

2.8.4 Example: Calcite

One example of a birefringent materialis calcite, which is also often used in optical devices, such as polarizers for example. Figure 2.117 and 2.118 show the arrangement of atoms in calcite.

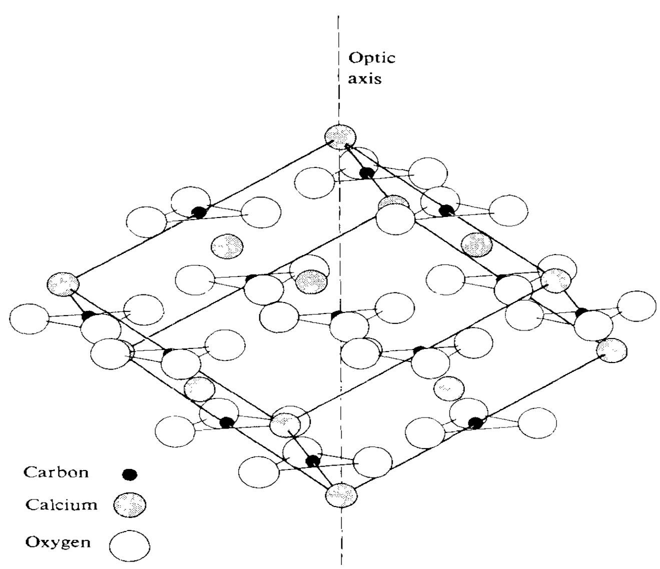

Figure 2.117: Arrangement of atoms in calcite, [1], p. 231.

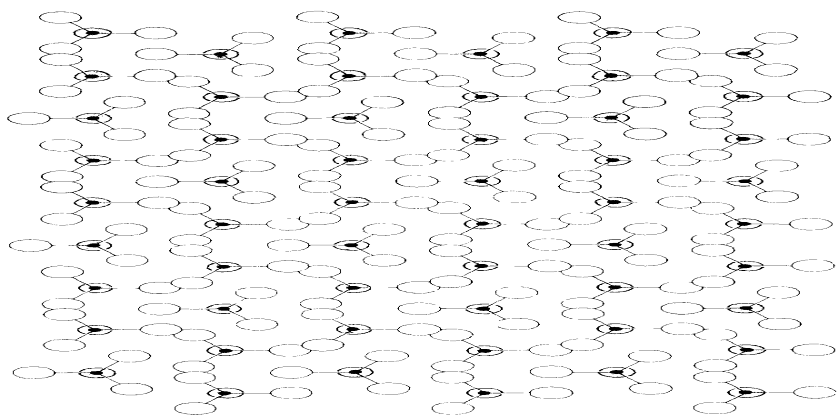

Figure 2.118: Atomic arrangement of calcite looking down the optical axis [1], p. 232.

Figure 2.119 shows a crystal cleaved along the crystal axis (cleavage form).

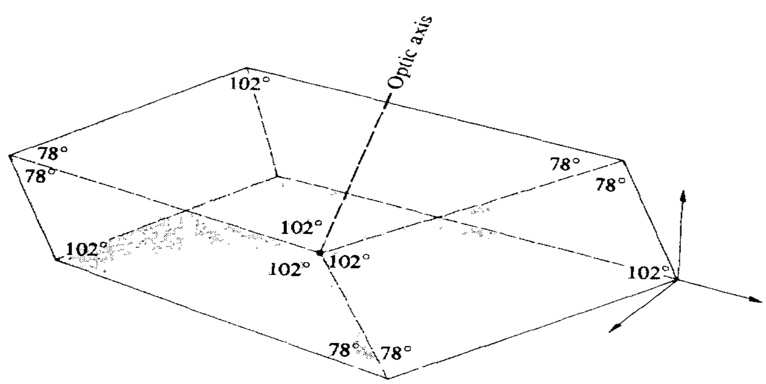

Figure 2.119: Calcite cleavage form [1], p. 232.

Figure 2.120 shows the light path of two orthogonally polarized light beams where one propagates as an ordinary and the other as an extraordinary wave through the crystal. This leads to a double image when an object is viewed through the crystal, see Figure 2.121.

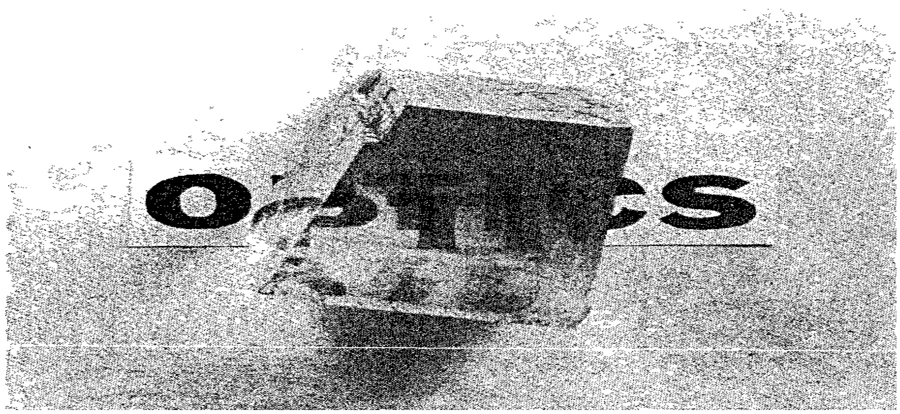

Figure 2.120: A light beam with two orthogonal field components traversing a calcite principal section [1], p. 234.

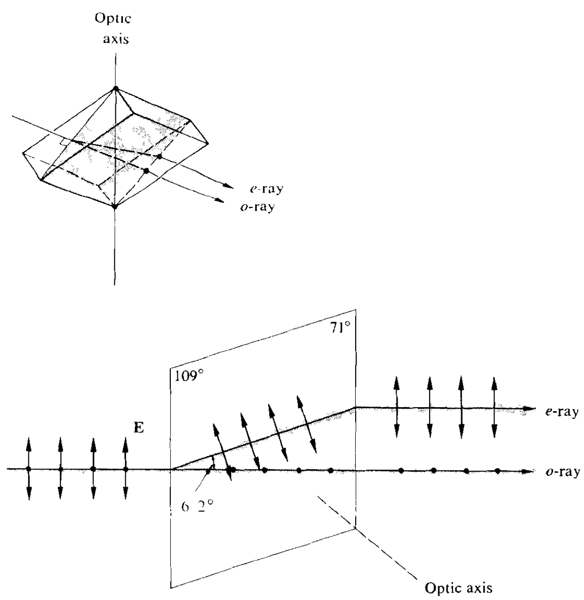

Figure 2.121: Double image formed by a calcite crystal (not cleavage form) [1], p. 233.

Table 2.8 gives the ordinary and extraordinary refractive indices of some uniaxial crystals. Birefringent materials enable the construction of wave

Crystal	n_o	n_e
Tourmaline	1.669	1.638
Calcite	1.6584	1.4864
Quartz	1.5443	1.5534
Sodim Nitrate	1.5854	1.3369
Ice	1.309	1.313
Rutile (TiO_2)	2.616	1.903

Table 2.8: Refractive indices of some uniaxial birefringent crystals ($\lambda =$ 589.3nm) [1], p.236

plates or retardation plates, which enable the manipulation of polarization in a very unique way.

2.9 Polarization and Crystal Optics

So far we have discussed linearly polarized electromagnetic waves, where the electric field of a TEM-wave propagating along the $z-$direction was either polarized along the $x-$ or $y-$axis. The most general TEM-wave has simultaneously electric fields in both polarizations and the direction of the electric field in space, i.e. its polarization, can change during propagation. A description of polarization and polarization evolution in optical systems can be based using Jones vectors and matrices.

2.9.1 Polarization

A general complex TEM-wave propagating along the $z-$direction is given by

$$\underline{\vec{E}}(z,t) = \begin{pmatrix} \underline{E}_{0x} \\ \underline{E}_{0y} \\ 0 \end{pmatrix} e^{\mathrm{j}(\omega t - kz)}, \tag{2.420}$$

where $\underline{E}_{0x} = E_{0x}e^{j\varphi_x}$ and $\underline{E}_{0y} = E_{0y}e^{j\varphi_y}$ are the complex field amplitudes of the $x-$ and $y-$ polarized components of the wave. The real electric field is given by

$$\vec{E}(z,t) = \begin{pmatrix} E_{0x}\cos\left(\omega t - kz + \varphi_x\right) \\ E_{0y}\cos\left(\omega t - kz + \varphi_y\right) \\ 0 \end{pmatrix}, \tag{2.421}$$

Both components are periodic functions in $\omega t - kz = \omega\left(t - z/c\right)$.

Linear Polarization

If the phases of the complex field amplitudes along the $x-$ and $y-$axis are equal, i.e.

$$\underline{E}_{0x} = \left|\underline{E}_{0x}\right| e^{j\varphi} \text{ and } \underline{E}_{0y} = \left|\underline{E}_{0y}\right| e^{j\varphi}$$

then the real electric field

$$\vec{E}(z,t) = \begin{pmatrix} \underline{E}_{0x} \\ \underline{E}_{0y} \\ 0 \end{pmatrix} \cos\left(\omega t - kz + \varphi\right) \tag{2.422}$$

always oscillates along a fixed direction in the x-y-plane, see Figure 2.122

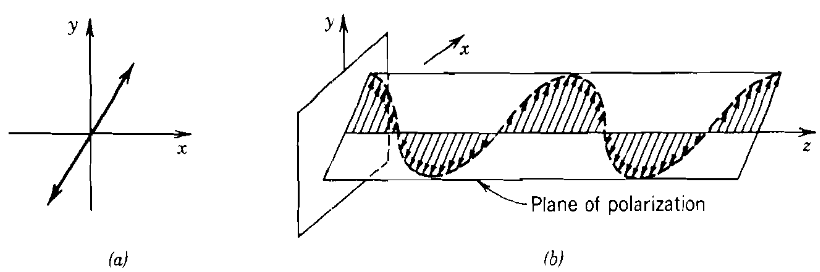

Figure 2.122: Linearly polarized light. (a) Time course at a fixed position z. (b) A snapshot at a fixed time t, [6], p. 197.

The angle between the polarization direction and the x-axis, α, is given by $\alpha = \arctan\left(E_{0y}/E_{0x}\right)$. If there is a phase difference of the complex field amplitudes along the $x-$ and $y-$axis, the direction and magnitude of the electric field amplitude changes periodically in time at a given position z.

Circular Polarization

Special cases occur when the magnitude of the fields in both linear polarizations are equal $E_{0x} = E_{0y} = E_0$, but there is a phase difference $\Delta\varphi = \pm\frac{\pi}{2}$ in both components. Then we obtain

$$\vec{E}(z,t) = E_0 \,\mathrm{Re}\left\{ \begin{pmatrix} e^{j\varphi} \\ e^{j(\varphi-\Delta\varphi)} \\ 0 \end{pmatrix} e^{\mathrm{j}(\omega t - kz)} \right\} \tag{2.423}$$

$$= E_0 \begin{pmatrix} \cos(\omega t - kz + \varphi) \\ \sin(\omega t - kz + \varphi) \\ 0 \end{pmatrix}. \tag{2.424}$$

For this case, the tip of the electric field vector describes a circle in the $x-y-$plane, as

$$|E_x(z,t)|^2 + |E_y(z,t)|^2 = E_0^2 \text{ for all } z, t, \tag{2.425}$$

see Figure 2.123.

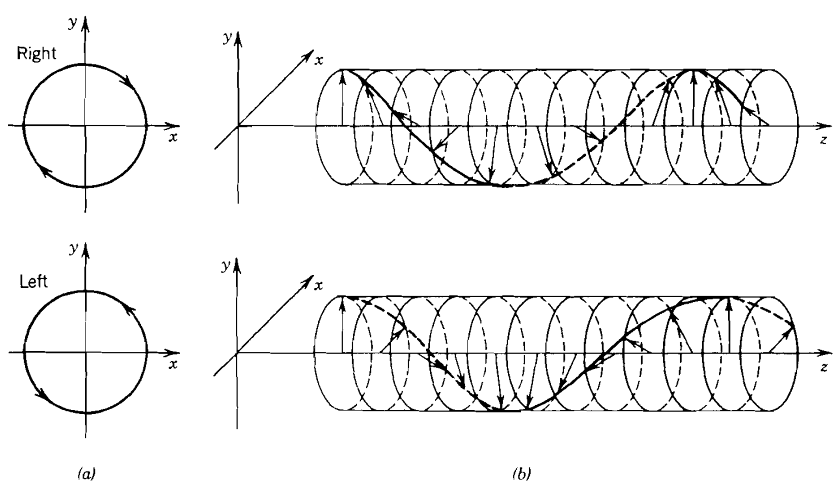

Figure 2.123: Trajectories of the tip of the electric field vector of a right and left circularly polarized plane wave. (a) Time course at a fixed position z. (b) A snapshot at a fixed time t. Note, the sense of rotation in (a) is opposite to that in (b) [6], p. 197.

Right Circular Polarization If the tip of the electric field at a given time, t, rotates counter clockwise with respect to the phase fronts of the wave, here in the positive $z-$direction, then the wave is called right circularly polarized light, i.e.

$$\vec{E}_{rc}(z,t) = E_0 \,\mathrm{Re}\left\{ \begin{pmatrix} 1 \\ j \\ 0 \end{pmatrix} e^{\mathrm{j}(\omega t - kz + \varphi)} \right\} = E_0 \begin{pmatrix} \cos(\omega t - kz + \varphi) \\ -\sin(\omega t - kz + \varphi) \\ 0 \end{pmatrix}. \tag{2.426}$$

A snapshot of the lines traced by the end points of the electric-field vectors at different positions is a right-handed helix, like a right-handed screw pointing in the direction of the phase fronts of the wave, i.e. $k-$vector see Figure 2.123 (b).

Left Circular Polarization If the tip of the electric field at a given fixed time, t, rotates clockwise with respect to the phase fronts of the wave, here in the again in the positive $z-$direction, then the wave is called left circularly polarized light, i.e.

$$\vec{E}_{lc}(z,t) = E_0 \,\mathrm{Re}\left\{ \begin{pmatrix} 1 \\ -j \\ 0 \end{pmatrix} e^{\mathrm{j}(\omega t - kz + \varphi)} \right\} = E_0 \begin{pmatrix} \cos(\omega t - kz + \varphi) \\ \sin(\omega t - kz + \varphi) \\ 0 \end{pmatrix}. \tag{2.427}$$

Eliptical Polarization The general polarization case is called eliptical polarization, as for arbitrary $\underline{E}_{0x} = E_{0x}e^{j\varphi_x}$ and $\underline{E}_{0y} = E_{0y}e^{j\varphi_y}$, we obtain for the locus of the tip of the electric field vector from

$$\vec{E}(z,t) = \begin{pmatrix} E_{0\times} \cos(\omega t - kz + \varphi_x) \\ E_{0y} \cos\left(\omega t - kz + \varphi_y\right) \\ 0 \end{pmatrix}. \tag{2.428}$$

the relations

$$\frac{E_y}{E_{0y}} = \cos\left(\omega t - kz + \varphi_y\right) \tag{2.429}$$

$$= \cos(\omega t - kz + \varphi_x)\cos\left(\varphi_y - \varphi_x\right) - \sin(\omega t - kz + \varphi_x)\sin\left(\varphi_y - \varphi_x\right). \tag{2.430}$$

and

$$\frac{E_x}{E_{0x}} = \cos\left(\omega t - kz + \varphi_x\right). \tag{2.431}$$

These relations can be combined to

$$\frac{E_y}{E_{0y}} - \frac{E_x}{E_{0x}}\cos\left(\varphi_y - \varphi_x\right) = -\sin\left(\omega t - kz + \varphi_x\right)\sin\left(\varphi_y - \varphi_x\right) \tag{2.432}$$

$$\sin\left(\omega t - kz + \varphi_x\right) = \sqrt{1 - \left(\frac{E_x}{E_{0x}}\right)^2} \tag{2.433}$$

Substituting Eq.(2.433) in Eq.(2.432) and building the square results in

$$\left(\frac{E_y}{E_{0y}} - \frac{E_x}{E_{0x}}\cos\left(\varphi_y - \varphi_x\right)\right)^2 = \left(1 - \left(\frac{E_x}{E_{0x}}\right)^2\right)\sin^2\left(\varphi_y - \varphi_x\right). \tag{2.434}$$

After reordering of the terms we obtain

$$\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 - 2\frac{E_x}{E_{0x}}\frac{E_y}{E_{0y}}\cos\left(\varphi_y - \varphi_x\right) = \sin^2\left(\varphi_y - \varphi_x\right). \tag{2.435}$$

This is the equation of an ellipse making an angle α with respect to the x-axis given by

$$\tan 2\alpha = \frac{2E_{0x}E_{0y}\cos\left(\varphi_y - \varphi_x\right)}{E_{0x}^2 - E_{0y}^2}. \tag{2.436}$$

see Figure 2.124.

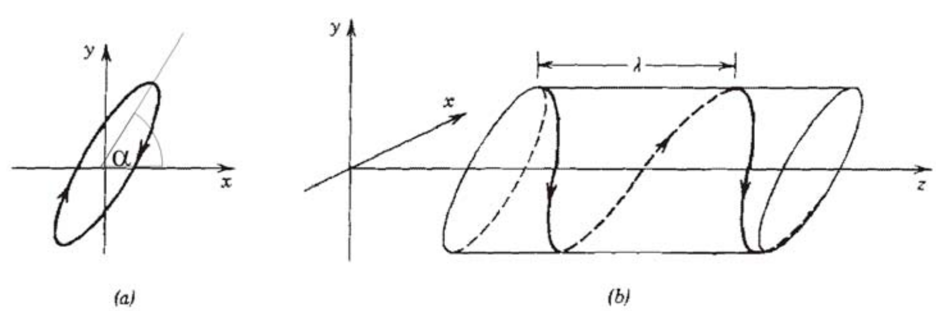

Figure 2.124: (a) Rotation of the endpoint of the electric field vector in the x-y-plane at a fixed position z. (b) A snapshot at a fixed time t [6], p. 197.

Elliptically polarized light can also be understood as a superposition of a right and left cicular polarized light, see Figure 2.125.

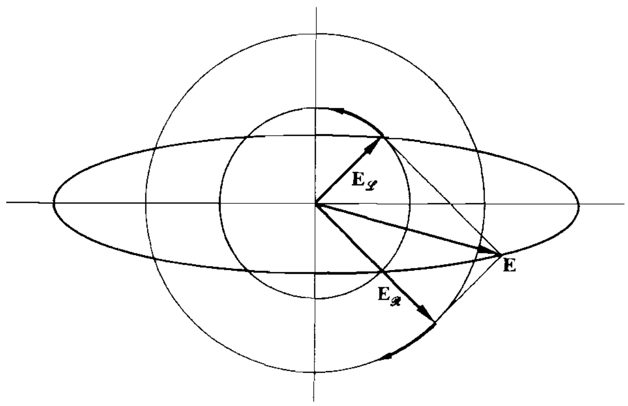

Figure 2.125: Elliptically polarized light as a superposition of right and left circularly polarized light [1], p. 223.

2.9.2 Jones Calculus

As seen in the last section, the information about polarization of a TEM-wave can be tracked by a vector that is proportional to the complex electric-field vector. This vector is called the Jones vector

$$\begin{pmatrix} \underline{E}_{0x} \\ \underline{E}_{0y} \end{pmatrix} \sim \underline{\vec{V}} = \begin{pmatrix} \underline{V}_x \\ \underline{V}_y \end{pmatrix} : \text{Jones Vector} \tag{2.437}$$

Jones Matrix

Figure 2.126 shows a light beam that is normally incident on a retardation plate along the $z-$axis with a polarization state described by a Jones vector

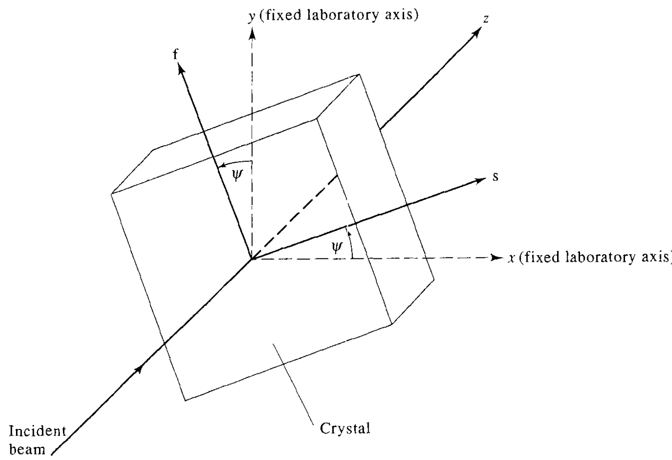

Figure 2.126: A retardation plate rotated at an angle ψ about the z-axis. f("fast") and s("slow") are the two principal dielectric axes of the crystal for light propagating along the $z-$axis [2], p. 17.

The principle axis ($s-$ for slow and $f-$ for fast axis) of the retardation plate are rotated by an angle ψ with respect to the $x-$ and $y-$axis. Let n_s and n_f be the refractive index of the slow and fast principle axis, respectively. The polarization state of the emerging beam in the crystal coordinate system is thus given by

$$\begin{pmatrix} V_s' \\ V_f' \end{pmatrix} = \begin{pmatrix} e^{-jk_o n_s L} & 0 \\ 0 & e^{-jk_o n_f L} \end{pmatrix} \begin{pmatrix} V_s \\ V_f \end{pmatrix}, \tag{2.438}$$

The phase retardation is defined as the phase difference between the two components

$$\Gamma = (n_s - n_f)\, k_o L. \tag{2.439}$$

In birefringent crystals the difference in refractive index is much smaller than the index itself, $|n_s - n_f| \ll n_s, n_f$, therefore parallel to the evolving differential phase a large absolute phase shift occurs. Taking the mean phase shift

$$\phi = \frac{1}{2}(n_s + n_f)\, k_o L, \tag{2.440}$$

out, we can rewrite (2.438) as

$$\begin{pmatrix} V_s' \\ V_f' \end{pmatrix} = e^{-j\phi} \begin{pmatrix} e^{-j\Gamma/2} & 0 \\ 0 & e^{j\Gamma/2} \end{pmatrix} \begin{pmatrix} V_s \\ V_f \end{pmatrix}. \tag{2.441}$$

The matrix connecting the Jones vector at the input of an optical component with the Jones vector at the output is called a Jones matrix.

If no coherent additon with another field is planned at the output of the system, the average phase ϕ can be dropped. With the rotation matrix, R, connecting the (x, y) coordinate system with the (s, f) coordinate system

$$R(\psi) = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}, \tag{2.442}$$

we find the Jones matrix W describing the propagation of the field components through the retardation plate as

$$\begin{pmatrix} V'_x \\ V'_y \end{pmatrix} = W \begin{pmatrix} V_x \\ V_y \end{pmatrix}. \tag{2.443}$$

with

$$W = R(-\psi)\, W_0 R(\psi). \tag{2.444}$$

and

$$W_0 = \begin{pmatrix} e^{-j\Gamma/2} & 0 \\ 0 & e^{j\Gamma/2} \end{pmatrix}. \tag{2.445}$$

Carrying out the matix multiplications leads to

$$W = \begin{pmatrix} e^{-j\Gamma/2}\cos^2(\psi) + e^{j\Gamma/2}\sin^2(\psi) & -j\sin\frac{\Gamma}{2}\sin(2\psi) \\ -j\sin\frac{\Gamma}{2}\sin(2\psi) & e^{-j\Gamma/2}\sin^2(\psi) + e^{j\Gamma/2}\cos^2(\psi) \end{pmatrix}. \tag{2.446}$$

Note that the Jones matrix of a wave plate is a unitary matrix, that is

$$W^{\dagger}W = 1.$$

Unitary matrices have the property that they transform orthogonal vectors into another pair of orthogonal vectors. Thus two orthogonal polarization states remain orthogonal when propagating through wave plates.

Polarizer

A polarizer is a device that absorbs one component of the polarization vector. The Jones matrix of polarizer along the x-axis or y-axis is

$$P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.447}$$

Half-Wave Plate

A half-wave plate has a phase retardation of $\Gamma = \pi$, i.e. its thickness is $t = \lambda/2(n_e - n_o)$. The corresponding Jones matrix follows from Eq.(2.446)

$$W = -j \begin{pmatrix} \cos(2\psi) & \sin(2\psi) \\ \sin(2\psi) & -\cos(2\psi) \end{pmatrix}. \quad (2.448)$$

For the special case of $\psi = 45^o$, see Figure 2.127, the half-wave plate rotates a linearly polarized beam exactly by 90^0, i.e. it exchanges the polarization axis. It can be shown, that for a general azimuth angle ψ, the half-wave plate will rotate the polarization by an angle 2ψ, see problem set. When the incident light is circularly polarized a half-wave plate will convert right-hand circularly polarized light into left-hand circularly polarized light and vice versa, regardless of the azimuth angle ψ.

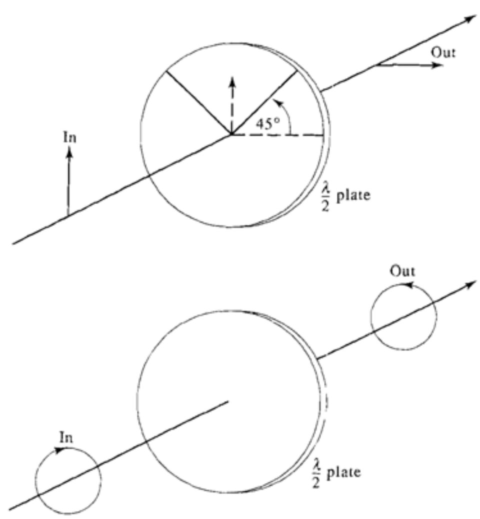

Figure 2.127: The effect of a half-wave plate on the polarziation state of a beam, [2], p.21.

Quarter-Wave Plate

A quarter-wave plate has a phase retardation of $\Gamma = \pi/2$, i.e. its thickness is $t = \lambda/4(n_e - n_o)$. The corresponding Jones matrix follows again from Eq.(2.446)

$$W = \begin{pmatrix} \frac{1}{\sqrt{2}}\left[1 - j\cos(2\psi)\right] & -j\frac{1}{\sqrt{2}}\sin(2\psi) \\ -j\frac{1}{\sqrt{2}}\sin(2\psi) & \frac{1}{\sqrt{2}}\left[1 + j\cos(2\psi)\right] \end{pmatrix}. \tag{2.449}$$

and for the special case of $\psi = 45^o$, see Figure 2.127 we obtain

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -j \\ -j & 1 \end{pmatrix}, \tag{2.450}$$

see Figure 2.128.

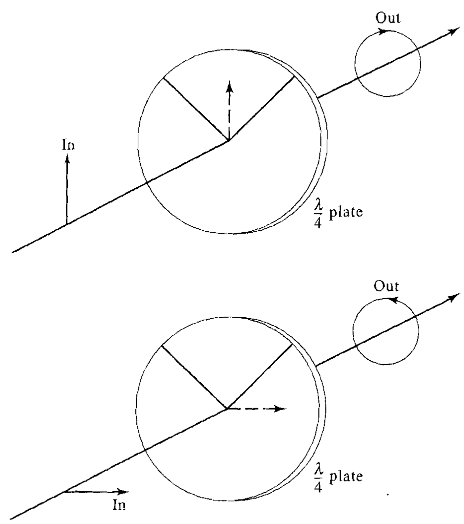

Figure 2.128: The effect of a quarter wave plate on the polarization state of a linearly polarized input wave [2], p.22.

If the incident beam is vertically polarized, i.e.

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.451}$$

the effect of a 45^o -oriented quarter-wave plate is to convert vertically polarized light into left-handed circularly polarized light. If the incident beam is horizontally polarized the outgoing beam is a right-handed circularly polarized, see Figure 2.128.

$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \frac{-j}{\sqrt{2}} \begin{pmatrix} 1 \\ j \end{pmatrix}. \tag{2.452}$$

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