### 5.8.3 Minimum Uncertainty States or Coherent States

From the matrix elements calculated in the last section, we find that the energy or quantum number eigenstates $|n\rangle$ have vanishing expected values for position and momentum. This also follows from the x-representation $\psi_n(x) = \langle x\,|n\rangle$ studied in section 4.4.2

$$\langle n|\,\mathbf{X}\,|n\rangle = 0\,, \qquad \langle n|\,\mathbf{P}\,|n\rangle = 0\,, \tag{5.168}$$

and the fluctuations in position and momentum are then simply

$$\langle n|\,\mathbf{X}^2\,|n\rangle = n + \frac{1}{2}\;, \qquad \langle n|\,\mathbf{P}^2\,|n\rangle = n + \frac{1}{2}\,. \tag{5.169}$$

The minimum uncertainty product for the fluctuations

$$\Delta X = \sqrt{\langle n|\,\mathbf{X}^2\,|n\rangle - \langle n|\,\mathbf{X}\,|n\rangle^2} = n + \frac{1}{2}\;, \tag{5.170}$$

$$\Delta P = \sqrt{\langle n|\,\mathbf{P}^2\,|n\rangle - \langle n|\,\mathbf{P}\,|n\rangle^2} = n + \frac{1}{2}. \tag{5.171}$$

is then

$$\Delta X \cdot \Delta P = n + \frac{1}{2}\;. \tag{5.172}$$

Only the ground state $n = 0$ is a minimum uncertainty wave packet, since it satisfies the eigenvalue equation

$$\mathbf{a}\,|0\rangle = 0, \tag{5.173}$$

where

$$\mathbf{a} = \frac{1}{\sqrt{2}}\,(\mathbf{X} + \mathrm{j}\mathbf{P})\,, \tag{5.174}$$

see problem set 8. In fact we can show that every eigenstate to the annihilation operator

$$\mathbf{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle, \text{ for } \alpha\epsilon\mathbb{C} \tag{5.175}$$

is a minimum uncertainty state. We obtain for expected values of position or momentum in these states

$$\left\langle\alpha\right|\mathbf{a}\left|\alpha\right\rangle = \alpha, \qquad \left\langle\alpha\right|\mathbf{a}^{+}\left|\alpha\right\rangle = \alpha^{*}, \tag{5.176}$$

$$\left\langle\alpha\right|\mathbf{a}^{+}\mathbf{a}\left|\alpha\right\rangle = \left|\alpha\right|^{2}, \qquad \left\langle\alpha\right|\mathbf{a}\mathbf{a}^{+}\left|\alpha\right\rangle = \left(\left|\alpha\right|^{2}+1\right), \tag{5.177}$$

$$\left\langle\alpha\right|\mathbf{X}\left|\alpha\right\rangle = \frac{1}{\sqrt{2}}\left(\alpha+\alpha^{*}\right), \left\langle\alpha\right|\mathbf{P}\left|\alpha\right\rangle = \frac{\mathrm{j}}{\sqrt{2}}\left(\alpha-\alpha^{*}\right), \tag{5.178}$$

and for its squares

$$\left\langle\alpha\right|\mathbf{a}^{+}\mathbf{a}\left|\alpha\right\rangle = \left|\alpha\right|^{2}, \qquad \left\langle\alpha\right|\mathbf{a}\mathbf{a}^{+}\left|\alpha\right\rangle = \left(\left|\alpha\right|^{2}+1\right), \tag{5.179}$$

$$\left\langle\alpha\right|\mathbf{a}^{2}\left|\alpha\right\rangle = \alpha^{2}, \qquad \left\langle\alpha\right|\mathbf{a}^{+2}\left|\alpha\right\rangle = \alpha^{*2}, \tag{5.180}$$

$$\left\langle\alpha\right|\mathbf{X}^{2}\left|\alpha\right\rangle = \frac{1}{2}\left(\alpha^{2}+2\alpha^{*}\alpha+\alpha^{*2}+1\right) = \left\langle\alpha\right|\mathbf{X}\left|\alpha\right\rangle^{2}+\frac{1}{2}, \tag{5.181}$$

$$\left\langle\alpha\right|\mathbf{P}^{2}\left|\alpha\right\rangle = \frac{1}{2}\left(-\alpha^{2}+2\alpha^{*}\alpha-\alpha^{*2}+1\right) = \left\langle\alpha\right|\mathbf{P}\left|\alpha\right\rangle^{2}+\frac{1}{2}. \tag{5.182}$$

Thus the uncertainty product is at its minimum

$$\Delta X\cdot\Delta P = \frac{1}{2}\ \forall\ \alpha\epsilon\mathbb{C}. \tag{5.183}$$

In fact one can show that the statistics of a position or momentum measurement for a harmonic oscillator in this state follows a Gaussian satistics with the average and variance given by Eqs.(5.178), (5.181) and (5.182). This can be represented pictorially in a phase space diagram as shown in Figure 5.1

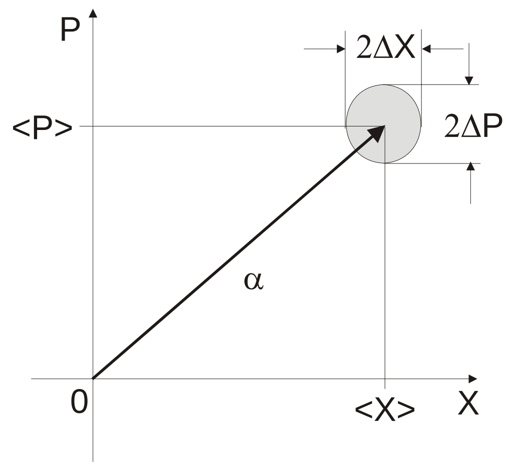


Figure 5.1: Representation of a minimum uncertainty state of the harmonic oscillator as a phase space distribution.

### 5.8.4 Heisenberg Picture

The Heisenberg equations of motion for a linear system like the harmonic oscillator are linear differential equations for the operators, which can be easily solved. From Eqs.(5.124) we find

$$\mathrm{j}\hbar \frac{\partial}{\partial t} \mathbf{a}_H(t) = [\mathbf{a}_H, \mathbf{H}] \tag{5.184}$$

$$= \hbar\omega_0 \mathbf{a}_H \ , \tag{5.185}$$

with the solution

$$\mathbf{a}_H(t) = e^{-\mathrm{j}\omega_0 t}\, \mathbf{a}_S \ . \tag{5.186}$$

Therefore, the expectation values for the creation, annihilation, position and momentum operators are identical to those of Eqs.(5.176) to (5.182); we only need to subsitute $\alpha \rightarrow \alpha e^{-\mathrm{j}\omega_0 t}$. We may again pictorially represent the time evolution of these states as a probability distribution in phase space, see Figure 5.2.

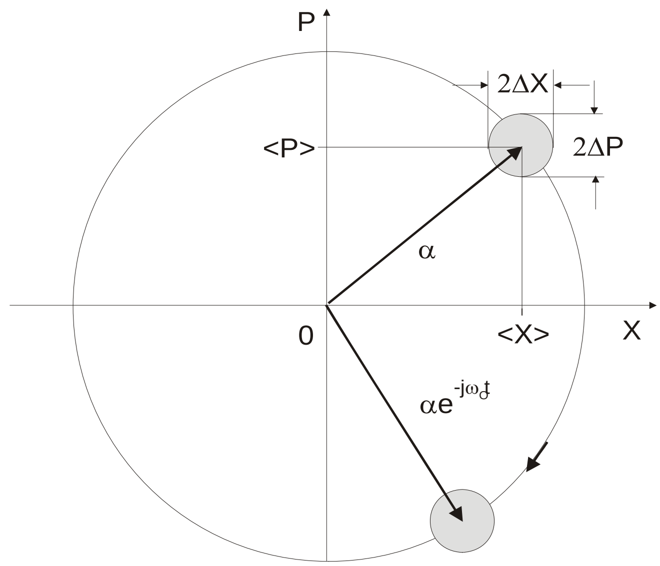


Figure 5.2: Time evolution of a coherent state in phase space.

## 5.9 The Kopenhagen Interpretation of Quantum Mechanics

### 5.9.1 Description of the State of a System

At a given time $t$ the state of a system is described by a normalized vector $|\Psi(t)\rangle$ in the Hilbert space, $H$. The Hilbert space is a linear vector space. Therefore, any linear combination of vectors is again a possible state of the system. Thus superpositions of states are possible and with it come interferences.

### 5.9.2 Description of Physical Quantities

Measurable physical quantities, observables, are described by hermitian operators $\mathbf{A} = \mathbf{A}^+$.

### 5.9.3 The Measurement of Observables

An observable has a spectral representation in terms of eigenvectors and eigenvalues, which can be discrete or continuous, here we discuss the discrete case

$$\mathbf{A} = \sum_n A_n \left| A_n \right\rangle \ \left\langle A_n \right| \ , \tag{5.187}$$

The eigenvectors are orthogonal to each other and the eigenvalues are real

$$\left\langle A_n \right| \ A_{n'} \rangle = \delta_{n,n'}. \tag{5.188}$$

Upon a measurement of the observable $\mathbf{A}$ of the system in state $|\Psi(t)\rangle$ the outcome can only be one of the eigenvalues $A_n$ of the observable and the probability for that event to occur is

$$p_n = \left| \left\langle A_n \right| \ \Psi(t) \rangle \right|^2. \tag{5.189}$$

If the eigenvalue spectrum of the operator $\mathbf{A}$ is degenerate, the probabilities of the probabilities of the different states to the same eigenvector need to be added.

After the measurement the system is in the eigenstate $|A_n\rangle$ corresponding to the eigenvalue $A_n$ found in the measurement, which is called the reduction of state[4]. This unphysical reduction of state is only necessary as a shortcut for the description of the measurement process and the fact that the system becomes entangled with the state of the macroscopic measurement equipment. This entanglement leads to a necessary decoherence of the superposition state of the measured system, which is equivalent to assuming a reduced state.

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