2.2 Electromagnetic Waves and Interfaces

Many microwave and optical devices are based on the characteristics of electromagnetic waves undergoing reflection or transmission at interfaces between media with different electric or magnetic properties characterized by ϵ and μ, see Fig. 2.17. Without restriction we can assume that the interface is the (x-y-plane) and the plane of incidence is the (x-z-plane). An arbitrary incident plane wave can always be decomposed into two components. One component has its electric field parallel to the interface between the media, i.e. it is polarized parallel to the interface and it is called the transverse electric (TE)-wave or also s-polarized wave. The other component is polarized in the plane of incidence and its magnetic field is in the plane of the interface between the media. This wave is called the TM-wave or also p-polarized wave. The most general case of an incident monochromatic TEM-wave is a linear superposition of a TE and a TM-wave.

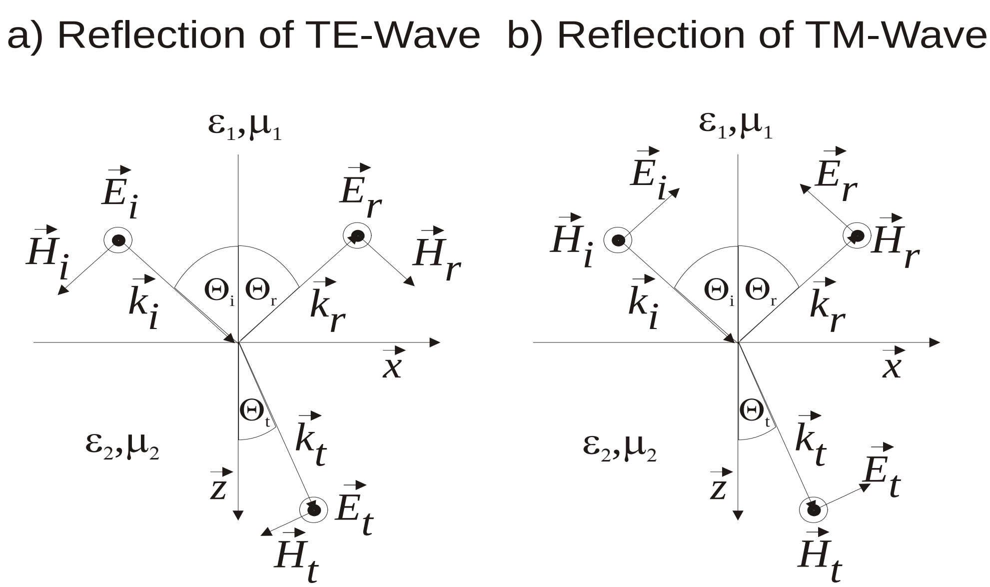

Figure 2.17: a) Reflection of a TE-wave at an interface, b) Reflection of aTM-wave at an interface

The fields for both cases are summarized in table 2.5

TE-wave	TM-wave
$\underline{\vec{E}}_i = \underline{E}_i\ e^{j(\omega t - \vec{k}_i \cdot \vec{r})} \vec{e}_y$	$\underline{\vec{E}}_i = -\underline{E}_i\ e^{j(\omega t - \vec{k}_i \cdot \vec{r})} \vec{e}_i$
$\underline{\vec{H}}_i = \underline{H}_i\ e^{j(\omega t - \vec{k}_i \cdot \vec{r})} \vec{h}_i$	$\underline{\vec{H}}_i = \underline{H}_i\ e^{j(\omega t - \vec{k}_i \cdot \vec{r})} \vec{e}_y$
$\underline{\vec{E}}_r = \underline{E}_r\ e^{j(\omega t - \vec{k}_r \cdot \vec{r})} \vec{e}_y$	$\underline{\vec{E}}_r = \underline{E}_r\ e_r^{j(\omega t - \vec{k}_r \cdot \vec{r})} \vec{e}_r$
$\underline{\vec{H}}_r = \underline{H}_r\ e^{j(\omega t - \vec{k}_r \cdot \vec{r})} \vec{h}_r$	$\underline{\vec{H}}_r = \underline{E}_r\ e^{j(\omega t - \vec{k}_r \cdot \vec{r})} \vec{e}_y$
$\underline{\vec{E}}_t = \underline{E}_t\ e^{j(\omega t - \vec{k}_t \cdot \vec{r})} \vec{e}_y$	$\underline{\vec{E}}_t = \underline{E}_t\ e^{j(\omega t - \vec{k}_t \cdot \vec{r})} \vec{e}_t$
$\underline{\vec{H}}_t = \underline{H}_t\ e^{j(\omega t - \vec{k}_t \cdot \vec{r})} \vec{h}_t$	$\underline{\vec{H}}_t = \underline{H}_t\ e^{j(\omega t - \vec{k}_t \cdot \vec{r})} \vec{e}_y$

Table 2.5: Electric and magnetic fields for TE- and TM-waves.

with wave vectors of the waves given by

$$\begin{aligned}
k_i &= k_r = k_0\sqrt{\epsilon_1 \mu_1}, \\
k_t &= k_0\sqrt{\epsilon_2 \mu_2}, \\
\vec{k}_{i,t} &= k_{i,t}\left(\sin\theta_{i,t}\ \vec{e}_x + \cos\theta_{i,t}\ \vec{e}_z\right), \\
\vec{k}_r &= k_i\left(\sin\theta_r\ \vec{e}_x - \cos\theta_r\ \vec{e}_z\right),
\end{aligned}$$

and unit vectors given by

$$\begin{aligned}
\vec{h}_{i,t} &= -\cos\theta_{i,t}\ \vec{e}_x + \sin\theta_{i,t}\ \vec{e}_z, \\
\vec{h}_r &= \cos\theta_r\ \vec{e}_x + \sin\theta_r\ \vec{e}_z, \\
\vec{e}_{i,t} &= -\vec{h}_{i,t} = \cos\theta_{i,t}\ \vec{e}_x - \sin\ \theta_{i,t}\ \vec{e}_z, \\
\vec{e}_r &= -\vec{h}_r = -\cos\theta_r\ \vec{e}_x - \sin\theta_r\ \vec{e}_z.
\end{aligned}$$

2.2.1 Boundary Conditions and Snell's law

From 6.013, we know that Stoke's and Gauss' Law for the electric and magnetic fields require constraints on some of the field components at media boundaries. In the absence of surface currents and charges, the tangential electric and magnetic fields as well as the normal dielectric and magnetic fluxes have to be continuous when going from medium 1 into medium 2 for all times at each point along the surface, i.e. $z = 0$

$$E/H_{i,x/y} e^{j(\omega t - k_{i,x} x)} + E/H_{r,x/y} e^{j(\omega t - k_{r,x} x)} = E/H_{i,x/y} e^{j(\omega t - k_{t,x} x)}. \tag{2.103}$$

This equation can only be fulfilled at all times if and only if the x-component of the k-vectors for the reflected and transmitted wave are equal to (match) the corresponding component of the incident wave

$$k_{i,x} = k_{r,x} = k_{t,x} \tag{2.104}$$

This phase matching condition is shown in Fig. 2.18 for the case $\sqrt{\epsilon_2\mu_2} > \sqrt{\epsilon_1\mu_1}$ *or* $k_t > k_i$.

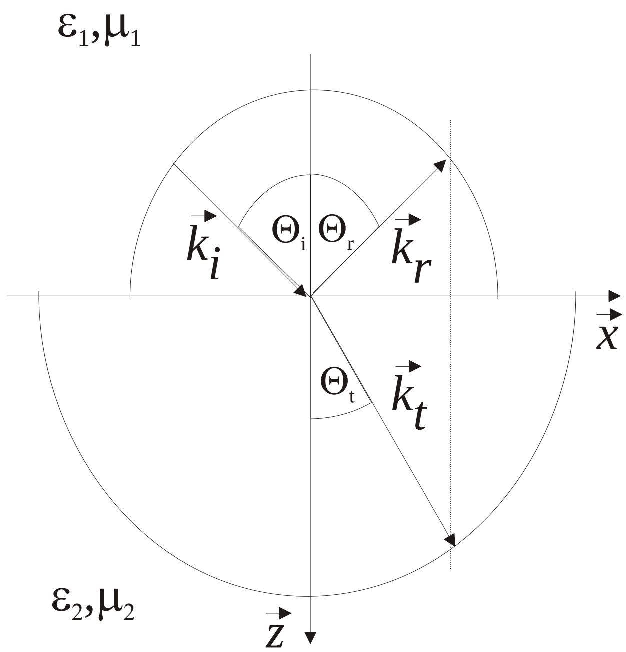

Figure 2.18: Phase matching condition for reflected and transmitted wave

The phase matching condition Eq(2.104) results in $\theta_r = \theta_i = \theta_1$ and Snell's law for the angle $\theta_t = \theta_2$ of the transmitted wave

$$\sin\theta_t = \frac{\sqrt{\epsilon_1\mu_1}}{\sqrt{\epsilon_2\mu_2}}\sin\theta_i \tag{2.105}$$

or for the case of non magnetic media with $\mu_1 = \mu_2 = \mu_0$

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i \tag{2.106}$$

2.2.2 Measuring Refractive Index with Minimum Deviation

Snell's law can be used for measuring the refractive index of materials. Consider a prism prepared from a material with unknown refractive index $n(\lambda)$, see Fig. 2.19 (a).

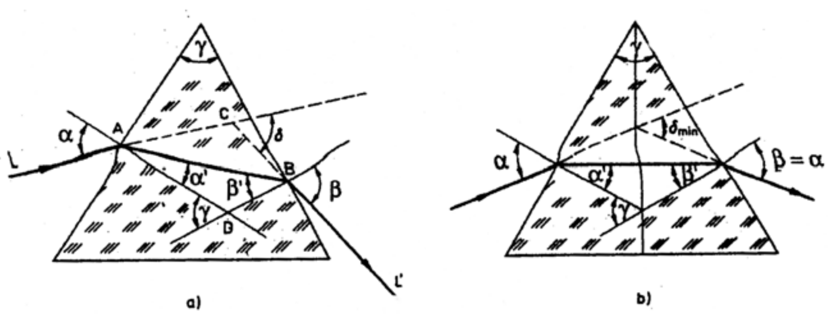

Figure 2.19: (a) Beam propagating through a prism. (b) For the case of minimum deviation [3] p. 65.

The prism is mounted on a rotation stage as shown in Fig. 2.20. The angle of incidence α is then varied with a fixed incident beam path and the transmitted light is observed on a screen. If one starts of with normal incidence on the first prism surface one notices that after turning the prism one goes through a minium for the deflection angle of the beam. This becomes obvious from Fig. 2.19 (b). There is an angle of incidence α where the beam path through the prism is symmetric. If the input angle is varied around this point, it would be identical to exchange the input and output beams. From that we conclude that the deviation δ must go through an extremum at the symmetry point, see Figure 2.21. It can be shown (Recitations), that the refractive index is then determined by

$$n = \frac{\sin \frac{\alpha(\delta_{\min})+\delta_{\min}}{2}}{\sin \frac{\alpha(\delta_{\min})}{2}}. \tag{2.107}$$

If the measurement is repeated for various wavelength of the incident radiation the complete wavelength dependent refractive index is characterized, see for example, Fig. 2.22.

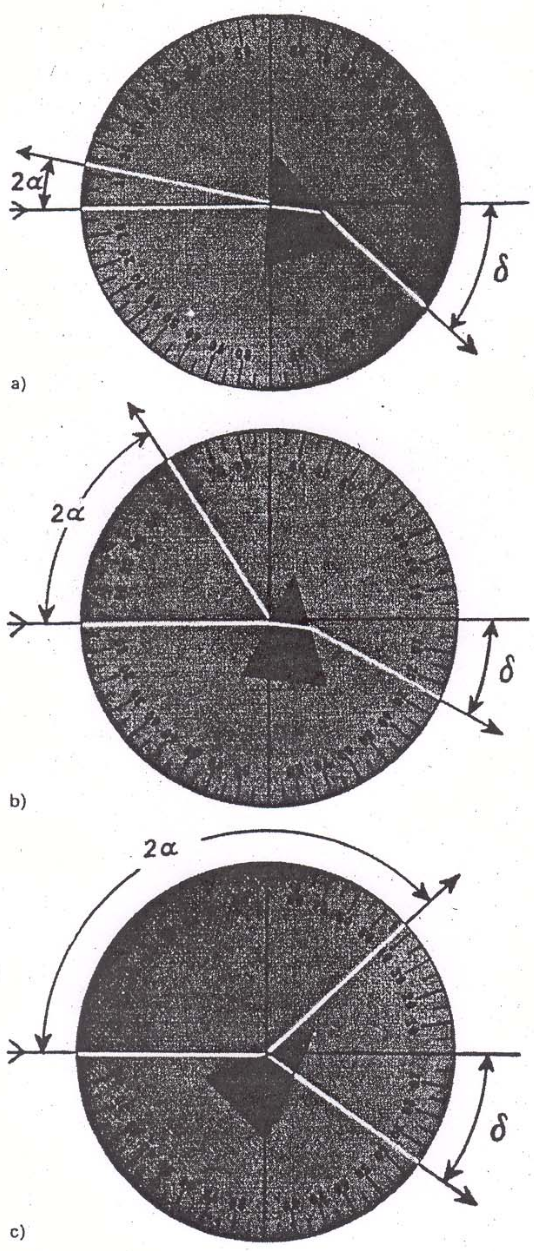

Figure 2.20: Refraction of a Prism with n=1.731 for different angles of incidence alpha. The angle of incidence is stepwise increased by rotating the prism clockwise. The angle of transmission first increases. After the angle for minimum deviation is reached the transmission angle starts to decrease [3] p67.

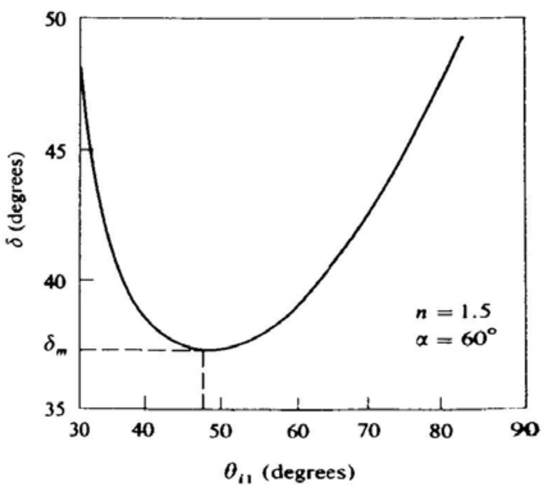

Figure 2.21: Deviation versus incident angel [1]

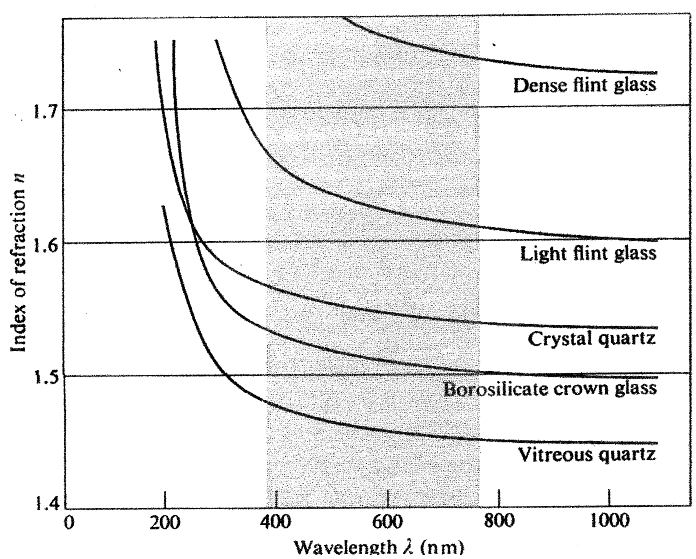

Figure 2.22: Refractive index as a function of wavelength for various media transmissive in the visible [1], p42.

2.2.3 Fresnel Reflection

After understanding the direction of the reflected and transmitted light, formulas for how much light is reflected and transmitted are derived by evaluating the boundary conditions for the TE and TM-wave. According to Eqs.(2.103) and (2.104) we obtain for the continuity of the tangential E and H fields:

TE-wave (s-pol.)	TM-wave (p-pol.)
$\underline{E}_i + \underline{E}_r = \underline{E}_t$	$\underline{E}_i \cos\theta_i - \underline{E}_r \cos\theta_r = \underline{E}_t \cos\theta_t$
$\underline{H}_i \cos\theta_i - \underline{H}_r \cos\theta_r = \underline{H}_t \cos\theta_t$	$\underline{H}_i + \underline{H}_r = \underline{H}_t$

(2.108)

Introducing the characteristic impedances in both half spaces $\underline{Z}_{1/2} = \sqrt{\frac{\mu_0 \mu_{1/2}}{\epsilon_0 \epsilon_{1/2}}}$, and the impedances that relate the tangential electric and magnetic field components $\underline{Z}_{1/2}^{TE/TM}$ in both half spaces the boundary conditions can be rewritten in terms of the electric or magnetic field components.

TE-wave (s-pol.)	TM-wave (p-pol.)
$\underline{Z}_{1/2}^{TE} = \frac{\underline{E}_{i/t}}{\underline{H}_{i/t} \cos\theta_{i/t}} = \frac{\underline{Z}_{1/2}}{\cos\theta_{1/2}}$	$\underline{Z}_{1/2}^{TM} = \frac{\underline{E}_{i/t} \cos\theta_{i/t}}{\underline{H}_{i/t}} = \underline{Z}_{1/2} \cos\theta_{1/2}$
$\underline{E}_i + \underline{E}_r = \underline{E}_t$	$\underline{H}_i - \underline{H}_r = \frac{\underline{Z}_2^{TM}}{\underline{Z}_1^{TM}} \underline{H}_t$
$\underline{E}_i - \underline{E}_r = \frac{\underline{Z}_1^{TE}}{\underline{Z}_2^{TE}} \underline{E}_t$	$\underline{H}_i + \underline{H}_r = \underline{H}_t$

(2.109)

Amplitude Reflection and Transmission coefficients

From these equations we can easily solve for the reflected and transmitted wave amplitudes in terms of the incident wave amplitudes. By dividing both equations by the incident wave amplitudes we obtain for the amplitude reflection and transmission coeffcients. Note, that reflection and transmission coefficients are defined in terms of the electric fields for the TE-wave and in terms of the magnetic fields for the TM-wave.

TE-wave (s-pol.)	TM-wave (p-pol.)
$\underline{r}^{TE} = \frac{\underline{E}_r}{\underline{E}_i}; \underline{t}^{TE} = \frac{\underline{E}_t}{\underline{E}_i}$	$\underline{r}^{TM} = \frac{\underline{H}_r}{\underline{H}_i}; \underline{t}^{TM} = \frac{\underline{H}_t}{\underline{H}_i}$
$1 + \underline{r}^{TE} = \underline{t}^{TE}$	$1 - \underline{r}^{TM} = \frac{\underline{Z}_2^{TM}}{\underline{Z}_1^{TM}} \underline{t}^{TM}$
$1 - \underline{r}^{TE} = \frac{\underline{Z}_1^{TE}}{\underline{Z}_2^{TE}} \underline{t}^{TE}$	$1 + \underline{r}^{TM} = \underline{t}^{TM}$

(2.110)

or in both cases the amplitude transmission and reflection coefficients are

$$\underline{t}^{TE/TM} = \frac{2}{1+\frac{\underline{Z}_{1/2}^{TE/TM}}{\underline{Z}_{2/1}^{TE/TM}}} = \frac{2\underline{Z}_{2/1}^{TE/TM}}{\underline{Z}_{1}^{TE/TM}+\underline{Z}_{2}^{TE/TM}} \tag{2.111}$$

$$\underline{r}^{TE/TM} = \frac{\underline{Z}_{2/1}^{TE/TM}-\underline{Z}_{1/2}^{TE/TM}}{\underline{Z}_{1}^{TE/TM}+\underline{Z}_{2}^{TE/TM}} \tag{2.112}$$

Despite the simplicity of these formulas, they describe already an enormous wealth of phenomena. To get some insight, consider the case of purely dielectric and lossless media characterized by its real refractive indices n_1 and n_2. Then Eqs.(2.111) and (2.112) simplify for the TE and TM case to

TE-wave (s-pol.)	TM-wave (p-pol.)
$\underline{Z}_{1/2}^{TE} = \frac{Z_{1/2}}{\cos\theta_{1/2}} = \frac{Z_0}{n_{1/2}\cos\theta_{1/2}}$	$\underline{Z}_{1/2}^{TM} = \underline{Z}_{1/2}\cos\theta_{1/2} = \frac{Z_0}{n_{1/2}}\cos\theta_{1/2}$
$\underline{r}^{TE} = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1+n_2\cos\theta_2}$	$\underline{r}^{TM} = \frac{\frac{n_2}{\cos\theta_2}-\frac{n_1}{\cos\theta_1}}{\frac{n_2}{\cos\theta_2}+\frac{n_1}{\cos\theta_1}}$
$\underline{t}^{TE} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1+n_2\cos\theta_2}$	$\underline{t}^{TM} = \frac{2\frac{n_2}{\cos\theta_2}}{\frac{n_2}{\cos\theta_2}+\frac{n_1}{\cos\theta_1}}$

(2.113)

Figure 2.23 shows the evaluation of Eqs.(2.113) for the case of a reflection at the interface of air and glass with $n_2 > n_1$ and ($n_1 = 1$, $n_2 = 1.5$).

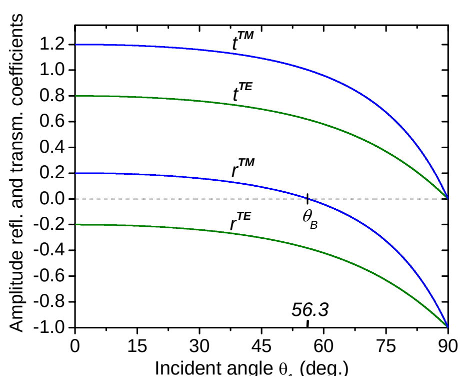

Figure 2.23: The amplitude coefficients of reflection and transmission as a function of incident angle. These correspond to external reflection $n_2 > n_1$ at an air-glas interface ($n_1 = 1$, $n_2 = 1.5$).

For TE-polarized light the reflected light changes sign with respect to the incident light (reflection at the optically more dense medium). This is not so for TM-polarized light under close to normal incidence. It occurs only for angles larger than θ_B, which is called the Brewster angle. So for TM-polarized light the amplitude reflection coefficient is zero at the Brewster angle. This phenomena will be discussed in more detail later.

This behavior changes drastically if we consider the opposite arrangement of media, i.e. we consider the glass-air interface with $n_1 > n_2$, see Figure 2.24. Then the TM-polarized light experiences a π-phase shift upon reflection close to normal incidence. For increasing angle of incidence this reflection coefficient goes through zero at the Brewster angle $\theta_B^{'}$ different from before. However, for large enough angle of incidence the reflection coefficient reaches magnitude 1 and stays there. This phenomenon is called total internal reflection and the angle where this occurs first is the critical angle for total internal reflection, θ_{tot}. Total internal reflection will be discussed in more detail later.

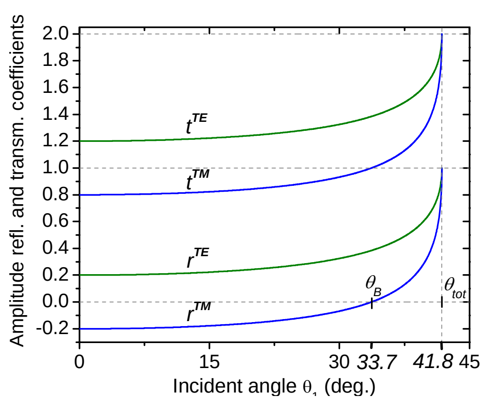

Figure 2.24: The amplitude coefficients of reflection and transmission as a function of incident angle. These correspond to internal reflection $n_1 > n_2$ at a glas-air interface ($n_1 = 1.5$, $n_2 = 1$).

Power reflection and transmission coefficients

Often we are not interested in the amplitude but rather in the optical power reflected or transmitted in a beam of finite size, see Figure 2.25.

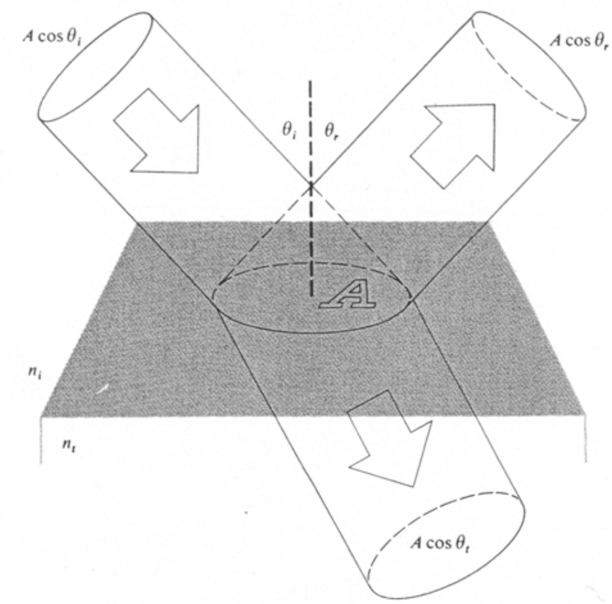

Figure 2.25: Reflection and transmission of an incident beam of finite size [1].

Note, that to get the power in a beam of finite size, we need to integrated the corresponding Poynting vector over the beam area, which means multiplication by the beam crosssectional area for a homogenous beam. Since the angle of incidence and reflection are equal, $\theta_i = \theta_r = \theta_1$ this beam crosssectional area drops out in reflection

$$R^{TE/TM} = \frac{I_r^{TE/TM} A\cos\theta_i}{I_i^{TE/TM} A\cos\theta_r} = \left|\underline{r}^{TE/TM}\right|^2 = \left|\frac{\underline{Z}_2^{TE/TM} - \underline{Z}_1^{TE/TM}}{\underline{Z}_1^{TE/TM} + \underline{Z}_2^{TE/TM}}\right|^2 \quad (2.114)$$

However, due to the different angles for the incident and the transmitted beam $\theta_t = \theta_2 \neq \theta_1$, we arrive at

$$\begin{aligned} T^{TE/TM} &= \frac{I_t^{TE/TM} A\cos\theta_t}{I_i^{TE/TM} A\cos\theta_r} \quad (2.115) \\ &= \frac{\cos\theta_2}{\cos\theta_1} \mathrm{Re}\left\{\frac{1}{\underline{Z}_{2/1}}\right\} \mathrm{Re}\left\{\frac{1}{\underline{Z}_{1/2}}\right\}^{-1} \left|\underline{t}^{TE/TM}\right|^2 . \end{aligned}$$

Using in the case of TE-polarization $\frac{\underline{Z}_{1/2}}{\cos\theta_{1/2}} = \underline{Z}_{1/2}^{TE}$ and analogously for TM-polarization $\underline{Z}_{1/2}\cos\theta_{1/2} = \underline{Z}_{1/2}^{TM}$, we obtain

$$T^{TE/TM} = \mathrm{Re}\left\{\frac{1}{\underline{Z}_{1/2}^{TE/TM}}\right\}^{-1} \mathrm{Re}\left\{\frac{4\underline{Z}_{2/1}^{TE/TM}}{\left|\underline{Z}_1^{TE/TM} + \underline{Z}_2^{TE/TM}\right|^2}\right\} \tag{2.116}$$

Note, for the case where the characteristic impedances are complex this can not be further simplified. If the characteristic impedances are real, i.e. the media are lossless, the transmission coefficient simplifies to

$$T^{TE/TM} = \frac{4Z_{1/2}^{TE/TM} Z_{2/1}^{TE/TM}}{\left(Z_1^{TE/TM} + Z_2^{TE/TM}\right)^2}. \tag{2.117}$$

To summarize for lossless media the power reflection and transmission coefficients are

TE-wave (s-pol.)	TM-wave (p-pol.)
$Z_{1/2}^{TE} = \frac{Z_{1/2}}{\cos\theta_{1/2}} = \frac{Z_0}{n_{1/2}\cos\theta_{1/2}}$	$Z_{1/2}^{TM} = Z_{1/2}\cos\theta_{1/2} = \frac{Z_0}{n_{1/2}}\cos\theta_{1/2}$
$R^{TE} = \left\|\frac{n_2\cos\theta_2 - n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2}\right\|^2$	$R^{TM} = \left\|\frac{\frac{n_2}{\cos\theta_2} - \frac{n_1}{\cos\theta_1}}{\frac{n_2}{\cos\theta_2} + \frac{n_1}{\cos\theta_1}}\right\|^2$
$T^{TE} = \frac{4n_1\cos\theta_1 n_2\cos\theta_2}{\|n_1\cos\theta_1 + n_2\cos\theta_2\|^2}$	$T^{TM} = \frac{4\frac{n_2}{\cos\theta_2}\frac{n_1}{\cos\theta_1}}{\left\|\frac{n_2}{\cos\theta_2} + \frac{n_1}{\cos\theta_1}\right\|^2}$
$T^{TE} + R^{TE} = 1$	$T^{TM} + R^{TM} = 1$

(2.118)

A few phenomena that occur upon reflection at surfaces between different media are especially noteworthy and need a more indepth discussion because they enhance or enable the construction of many optical components and devices.

2.2.4 Brewster's Angle

As Figures 2.23 and 2.24 already show, for light polarized parallel to the plane of incidence, p-polarized light, the reflection coefficient vanishes at a given angle θ_B, called the Brewster angle. Using Snell's Law Eq.(2.106),

$$\frac{n_2}{n_1} = \frac{\sin\theta_1}{\sin\theta_2}, \tag{2.119}$$

we can rewrite the reflection and transmission coefficients in Eq.(2.118) only in terms of the angles. For example, we find for the reflection coefficient

$$R^{TM} = \left| \frac{\frac{n_2}{n_1} - \frac{\cos\theta_2}{\cos\theta_1}}{\frac{n_2}{n_1} + \frac{\cos\theta_2}{\cos\theta_1}} \right|^2 = \left| \frac{\frac{\sin\theta_1}{\sin\theta_2} - \frac{\cos\theta_2}{\cos\theta_1}}{\frac{\sin\theta_1}{\sin\theta_2} + \frac{\cos\theta_2}{\cos\theta_1}} \right|^2 \tag{2.120}$$

$$= \left| \frac{\sin 2\theta_1 - \sin 2\theta_2}{\sin 2\theta_1 + \sin 2\theta_2} \right|^2 \tag{2.121}$$

where we used in the last step in addition the relation $\sin 2\alpha = 2\sin\alpha\cos\alpha$. Thus by forcing $R^{TM} = 0$, the Brewster angle is reached for

$$\sin 2\theta_{1,B} - \sin 2\theta_{2,B} = 0 \tag{2.122}$$

or

$$2\theta_{1,B} = \pi - 2\theta_{2,B} \text{ or } \theta_{1,B} + \theta_{2,B} = \frac{\pi}{2} \tag{2.123}$$

This relation is illustrated in Figure 2.26. The reflected and transmitted beams are orthogonal to each other, so that the dipoles induced in the medium by the transmitted beam, shown as arrows in Fig. 2.26, can not radiate into the direction of the reflected beam. This is the physical origin of the zero in the reflection coefficient, only possible for a p-polarized or TM-wave.

The relation (2.123) can be used to express the Brewster angle as a function of the refractive indices, because if we substitute (2.123) into Snell's law we obtain

$$\begin{aligned} \frac{\sin\theta_1}{\sin\theta_2} &= \frac{n_2}{n_1} \\ \frac{\sin\theta_{1,B}}{\sin\left(\frac{\pi}{2} - \theta_{1,B}\right)} &= \frac{\sin\theta_{1,B}}{\cos\theta_{1,B}} = \tan\theta_{1,B}, \end{aligned}$$

or

$$\tan\theta_{1,B} = \frac{n_2}{n_1}. \tag{2.124}$$

Using the Brewster angle condition one can insert an optical component with a refractive index $n \neq 1$ into a TM-polarized beam in air without having reflections, see Figure 2.27. Note, this is not possible for a TE-polarized beam.

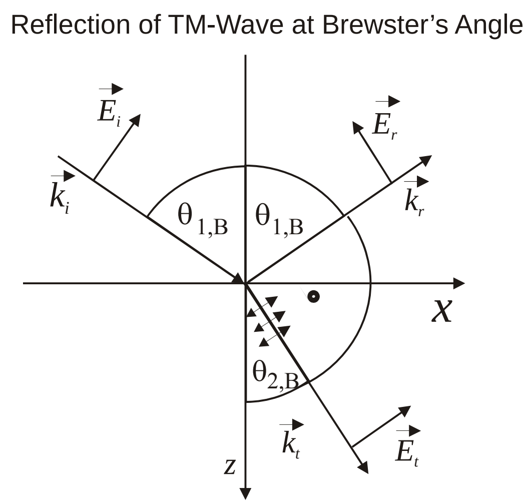

Figure 2.26: Conditions for reflection of a TM-Wave at Brewster's angle. The reflected and transmitted beams are orthogonal to each other, so that the dipoles excited in the medium by the transmitted beam can not radiate into the direction of the reflected beam.

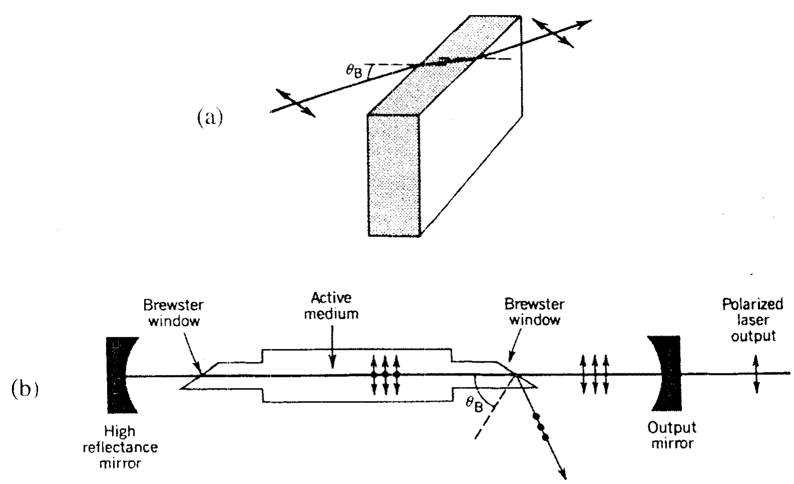

Figure 2.27: A plate under Brewster's angle does not reflect TM-light. The plate can be used as a window to introduce gas filled tubes into a laser beam without insertion loss (ideally), [6] p. 209.

2.2.5 Total Internal Reflection

Another striking phenomenon, see Figure 2.24, occurs for the case where the beam hits the surface from the side of the optically denser medium, i.e. $n_1 > n_2$. There is obviously a critical angle of incidence, beyond which all light is reflected. How can that occur? This is easy to understand from the phase matching diagram at the surface, see Figure 2.18, which is redrawn for this case in Figure 2.28.

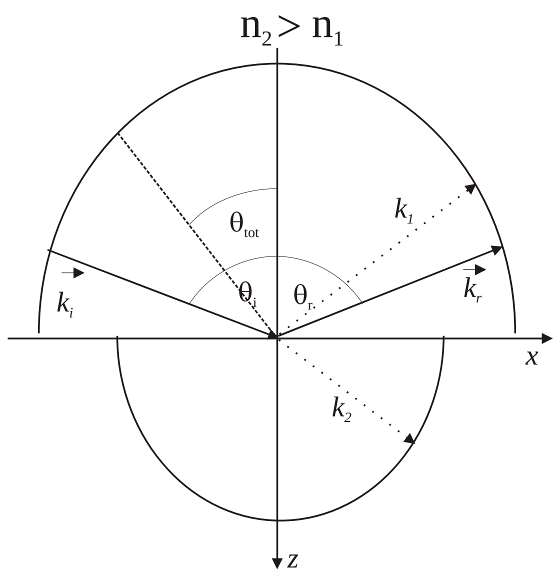

Figure 2.28: Phase matching diagram for total internal reflection.

There is no real wavenumber in medium 2 possible as soon as the angle of incidence becomes larger than the critical angle for total internal reflection

$$\theta_i > \theta_{tot} \tag{2.125}$$

with

$$\sin\theta_{tot} = \frac{n_2}{n_1}. \tag{2.126}$$

Figure 2.29 shows the angle of refraction and incidence for the two cases of external and internal reflection, when the angle of incidence approaches the critical angle.

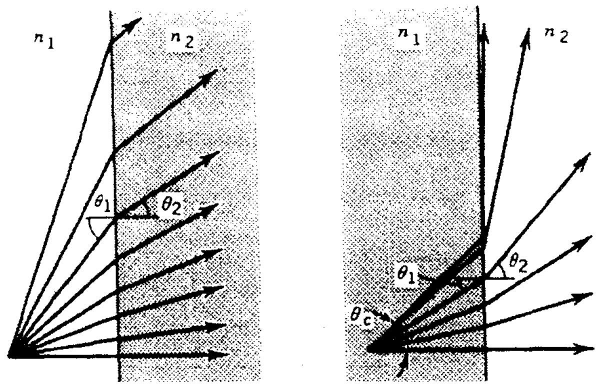

Figure 2.29: Relation between angle of refraction and incidence for external refraction and internal refraction ([6], p. 11).

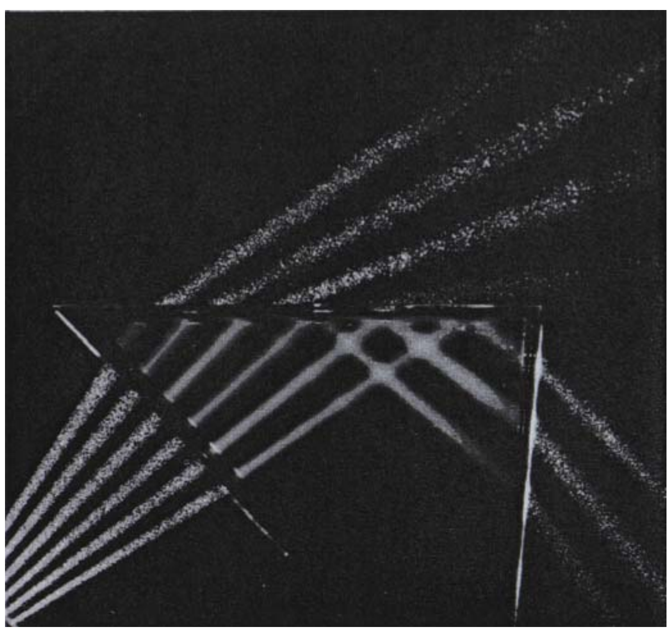

Figure 2.30: Relation between angle of refraction and incidence for external refraction and internal refraction ([1], p. 81).

Total internal reflection enables broadband reflectors. Figure 2.30 shows again what happens when the critical angle of reflection is surpassed. Figure 2.31 shows how total internal reflection can be used to guide light via reflection at a prism or by multiple reflections in a waveguide.

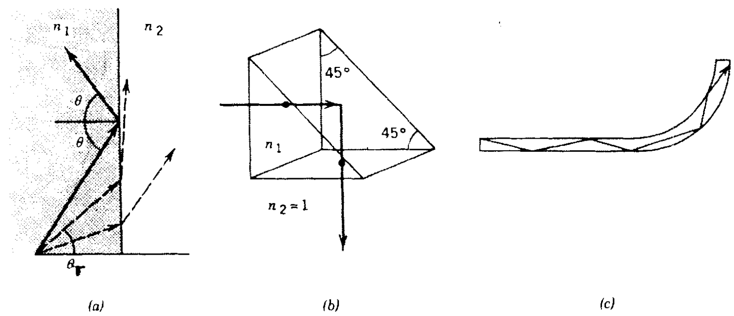

Figure 2.31: (a) Total internal reflection, (b) internal reflection in a prism, (c) Rays are guided by total internal reflection from the internal surface of an optical fiber ([6] p. 11).

Figure 2.32 shows the realization of a retro reflector, which always returns a parallel beam independent of the orientation of the prism (in fact the prism can be a real 3D-corner so that the beam is reflected parallel independent from the precise orientation of the corner cube). A surface patterned by little corner cubes constitute a "cats eye" used on traffic signs.

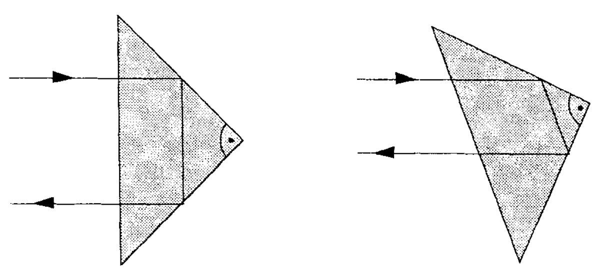

Figure 2.32: Total internal reflection in a retro reflector.

More on reflecting prisms and its use can be found in [1], pages 131-136.

Evanescent Waves

What is the field in medium 2 when total internal reflection occurs? Is it identical to zero? It turns out phase matching can still occur if the propagation constant in z-direction becomes imaginary, $k_{2z} = -\mathrm{j}\kappa_{2z}$, because then we can fulfill the wave equation in medium 2. This is equivalent to the dispersion relation

$$k_{2x}^2 + k_{2z}^2 = k_2^2,$$

or with $k_{2x} = k_{1x} = k_1 \sin\theta_1$, we obtain for the imaginary wavenumber

$$\begin{aligned} \kappa_{2z} &= \sqrt{k_1^2 \sin^2\theta_1 - k_2^2}, && (2.127)\\ &= k_1 \sqrt{\sin^2\theta_1 - \sin^2\theta_{tot}}. && (2.128) \end{aligned}$$

The electric field in medium 2 is then, for the example for a TE-wave, given by

$$\begin{aligned} \underline{\vec{E}}_t &= \underline{E}_t\ \vec{e}_y\ e^{\mathrm{j}(\omega t - \vec{k}_t \cdot \vec{r})}, && (2.129)\\ & \underline{E}_t\ \vec{e}_y\ e^{\mathrm{j}(\omega t - k_{2,x} x)} e^{-\kappa_{2z} z}. && (2.130) \end{aligned}$$

Thus the wave penetrates into medium 2 exponentially with a 1/e-depth δ, given by

$$\delta = \frac{1}{\kappa_{2z}} = \frac{1}{k_1 \sqrt{\sin^2\theta_1 - \sin^2\theta_{tot}}} \qquad (2.131)$$

Figure 2.33 shows the penetration depth as a function of angle of incidence for a silica/air interface and a silicon/air interface. The figure demonstrates that light from inside a semiconductor material with a relatively high index around n=3.5 is mostly captured in the semiconductor material (Problem of light extraction from light emitting diodes (LEDs)), see problem set 2.

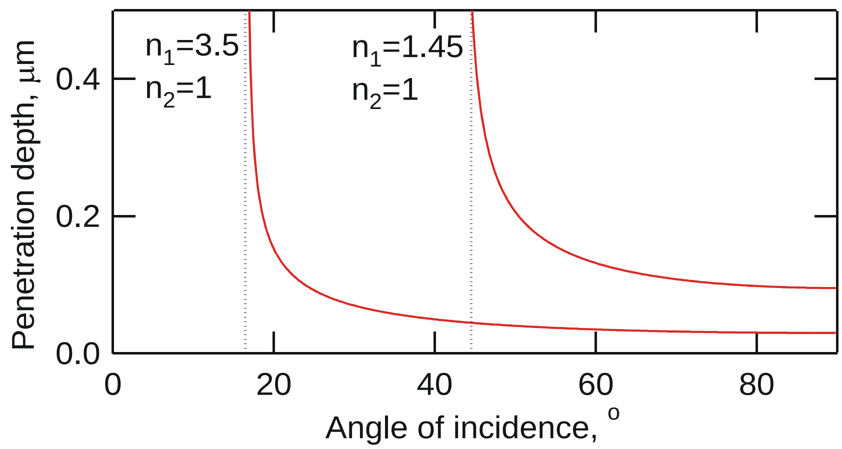

Figure 2.33: Penetration depth for total internal reflection at a silica/air and a silicon/air interface for $\lambda = 0.633$nm.

As the magnitude of the reflection coefficient is 1 for total internal reflection, the power flowing into medium 2 must vanish, i.e. the transmission is zero. Note, that the transmission and reflection coefficients in Eq.(2.113) can be used beyond the critical angle for total internal reflection. We only have to be aware that the electric field in medium 2 has an imaginary dependence in the exponent for the z-direction, i.e. $k_{2z} = k_2 \cos\theta_2 = -j\kappa_{2z}$. Thus $\cos\theta_2$ in all formulas for the reflection and transmission coefficients has to be replaced by the imaginary number

$$\begin{aligned} \cos\theta_2 &= \frac{k_{2z}}{k_2} = -\mathrm{j}\frac{k_1}{k_2}\sqrt{\sin^2\theta_1 - \sin^2\theta_{tot}} \\ &= -\mathrm{j}\frac{n_1}{n_2}\sqrt{\sin^2\theta_1 - \sin^2\theta_{tot}} \\ &= -\mathrm{j}\sqrt{\left(\frac{\sin\theta_1}{\sin\theta_{tot}}\right)^2 - 1}. \end{aligned} \tag{2.132}$$

Then the reflection coefficients in Eq.(2.113) change to all-pass functions

TE-wave (s-pol.)	TM-wave (p-pol.)
$\underline{r}^{TE} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$	$\underline{r}^{TM} = \frac{\frac{n_2}{\cos\theta_2} - \frac{n_1}{\cos\theta_1}}{\frac{n_2}{\cos\theta_2} + \frac{n_1}{\cos\theta_1}}$
$\underline{r}^{TE} = \frac{\cos\theta_1 + j\frac{n_2}{n_1}\sqrt{\left(\frac{\sin\theta_1}{\sin\theta_{tot}}\right)^2 - 1}}{\cos\theta_1 - j\frac{n_2}{n_1}\sqrt{\left(\frac{\sin\theta_1}{\sin\theta_{tot}}\right)^2 - 1}}$	$\underline{r}^{TM} = \frac{\cos\theta_1 + j\frac{n_1}{n_2}\sqrt{\left(\frac{\sin\theta_1}{\sin\theta_{tot}}\right)^2 - 1}}{\cos\theta_1 - j\frac{n_1}{n_2}\sqrt{\left(\frac{\sin\theta_1}{\sin\theta_{tot}}\right)^2 - 1}}$
$\tan\frac{\phi^{TE}}{2} = \frac{1}{\cos\theta_1}\frac{n_2}{n_1}\sqrt{\left(\frac{\sin\theta_1}{\sin\theta_{tot}}\right)^2 - 1}$	$\tan\frac{\phi^{TM}}{2} = \frac{1}{\cos\theta_1}\frac{n_1}{n_2}\sqrt{\left(\frac{\sin\theta_1}{\sin\theta_{tot}}\right)^2 - 1}$

(2.133)

Thus the magnitude of the reflection coefficient is 1. However, there is a non-vanishing phase shift for the light field upon total internal reflection, denoted as ϕ^{TE} and ϕ^{TM} in the table above. Figure 2.34 shows these phase shifts for the glass/air interface and for both polarizations.

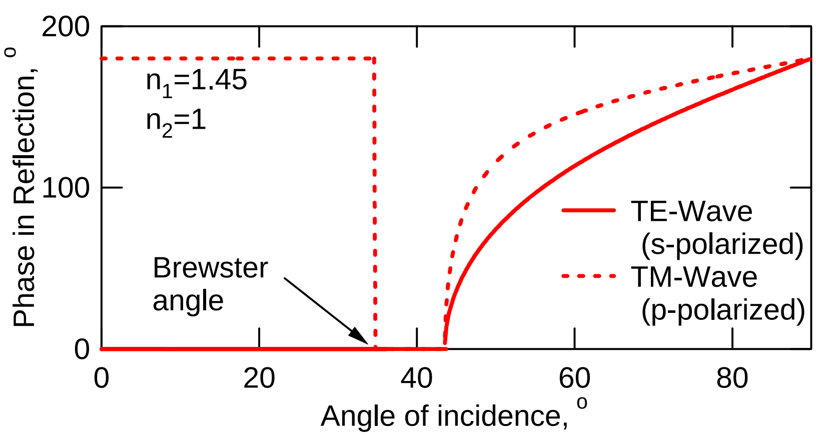

Figure 2.34: Phase shifts for TE- and TM- wave upon reflection from a silica/air interface, with $n_1 = 1.45$ and $n_2 = 1$.

Goos-Haenchen-Shift

So far, we looked only at plane waves undergoing reflection at surface due to total internal reflection. If a beam of finite transverse size is reflected from such a surface it turns out that it gets displaced by a distance Δz, see Figure 2.35 (a), called Goos-Haenchen-Shift.

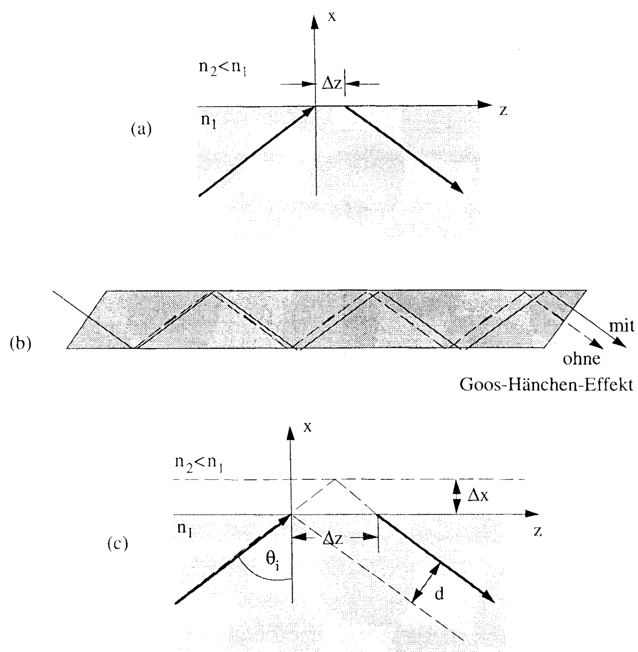

Figure 2.35: (a) Goos-Haenchen Shift and related beam displacement upon reflection of a beam with finite size; (b) Accumulation of phase shifts in a waveguide.

Detailed calculations show (problem set 2), that the displacement is given by

$$\Delta z = 2\delta^{TE/TM} \tan\theta_1, \tag{2.134}$$

as if the beam was reflected at a virtual layer with depth $\delta^{TE/TM}$ into medium 2. It turns out, that for TE-waves

$$\delta^{TE} = \delta, \tag{2.135}$$

where δ is the penetration depth according to Eq.(2.131) for evanescent waves. But for TM-waves

$$\delta^{TM} = \frac{\delta}{\left[1 + \left(\frac{n_1}{n_2}\right)^2\right] \sin^2\theta_1 - 1} \tag{2.136}$$

These shifts accumulate when the beam is propagating in a waveguide, see Figure 2.35 (b) and is important to understand the dispersion relations of waveguide modes. The Goos-Haenchen shift can be observed by reflection at a prism partially coated with a silver film, see Figure 2.36. The part reflected from the silver film is shifted with respect to the beam reflected due to total internal reflection, as shown in the figure.

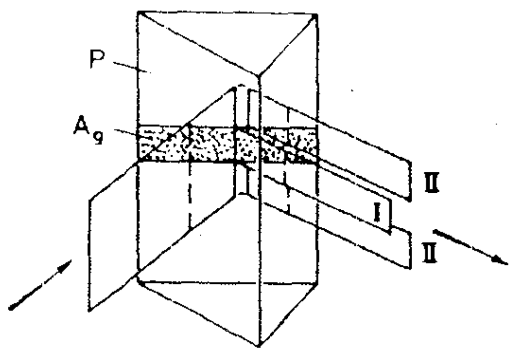

Figure 2.36: Experimental proof of the Goos-Haenchen shift by total internal reflection at a prism, that is partially coated with silver, where the penetration of light can be neglected. [3] p. 486.

Frustrated total internal reflection

Another proof for the penetration of light into medium 2 in the case of total internal reflection can be achieved by putting two prisms, where total internal reflection occurs back to back, see Figure 2.37. Then part of the light, depending on the distance between the two interfaces, is converted back into a propagating wave that can leave the second prism. This effect is called frustrated internal reflection and it can be used as a beam splitter as shown in Figure 2.37.

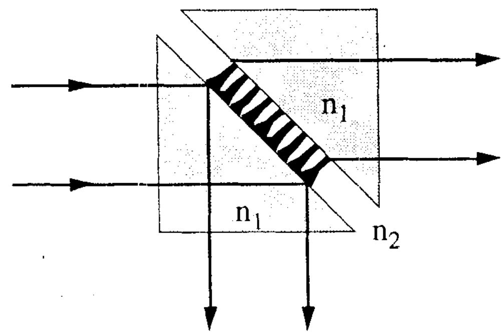

Figure 2.37: Frustrated total internal reflection. Part of the light is picked up by the second surface and converted into a propagating wave.