Chapter 9 Circular Motion Dynamics

Chapter 9 Equation Chapter 9 Section 1Circular Motion Dynamics

I shall now recall to mind that the motion of the heavenly bodies is circular, since the motion appropriate to a sphere is rotation in a circle.[1]

Nicholas Copernicus

9.1 Introduction Newton's Second Law and Circular Motion

We have already shown that when an object moves in a circular orbit of radius r with angular velocity $\vec{\boldsymbol{\omega}}$, it is most convenient to choose polar coordinates to describe the position, velocity and acceleration vectors. In particular, the acceleration vector is given by

$$\vec{\mathbf{a}}(t) = -r\left(\frac{d\theta}{dt}\right)^2 \hat{\mathbf{r}}(t) + r\frac{d^2\theta}{dt^2} \hat{\boldsymbol{\theta}}(t). \tag{9.1.1}$$

Then Newton's Second Law, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, can be decomposed into radial ($\hat{\mathbf{r}}$-) and tangential ($\hat{\boldsymbol{\theta}}$-) components

$$F_r = -mr\left(\frac{d\theta}{dt}\right)^2 \text{ (circular motion)}, \tag{9.1.2}$$

$$F_\theta = mr\frac{d^2\theta}{dt^2} \quad \text{(circular motion)}. \tag{9.1.3}$$

For the special case of uniform circular motion, $d^2\theta / dt^2 = 0$, and so the sum of the tangential components of the force acting on the object must therefore be zero,

$$F_\theta = 0 \quad \text{(uniform circular motion)}. \tag{9.1.4}$$

9.2 Universal Law of Gravitation and the Circular Orbit of the Moon

An important example of (approximate) circular motion is the orbit of the Moon around the Earth. We can approximately calculate the time T the Moon takes to complete one circle around the earth (a calculation of great importance to early lunar calendar systems, which became the basis for our current model.) Denote the distance from the moon to the center of the earth by $R_{e,m}$.

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[1] *Dedicatory Letter to Pope Paul III.*

Because the Moon moves nearly in a circular orbit with angular speed $\omega = 2\pi / T$ it is accelerating towards the Earth. The radial component of the acceleration (centripetal acceleration) is

$$a_r = -\frac{4\pi^2 R_{e,m}}{T^2}.$$ (9.2.1)

According to Newton's Second Law, there must be a centripetal force acting on the Moon directed towards the center of the Earth that accounts for this inward acceleration.

9.2.1 Universal Law of Gravitation

Newton's Universal Law of Gravitation describes the gravitational force between two bodies 1 and 2 with masses m_1 and m_2 respectively. This force is a radial force (always pointing along the radial line connecting the masses) and the magnitude is proportional to the inverse square of the distance that separates the bodies. Then the force on object 2 due to the gravitational interaction between the bodies is given by,

$$\vec{F}_{1,2} = -G\frac{m_1 m_2}{r_{1,2}^2}\hat{r}_{1,2},$$ (9.2.2)

where $r_{1,2}$ is the distance between the two bodies and $\hat{r}_{1,2}$ is the unit vector located at the position of object 2 and pointing from object 1 towards object 2. The Universal Gravitation Constant is $G = 6.67 \times 10^{-11}\,\text{N}\cdot\text{m}^2\cdot\text{kg}^{-2}$. Figure 9.1 shows the direction of the forces on bodies 1 and 2 along with the unit vector $\hat{r}_{1,2}$.
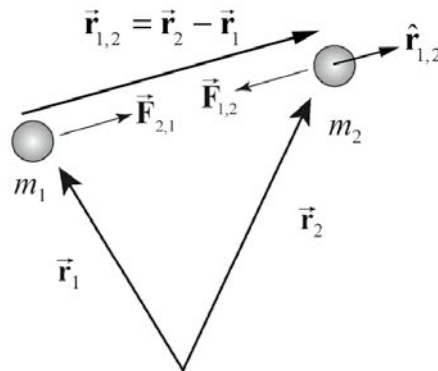

Figure 9.1 Gravitational force of interaction between two bodies

Newton realized that there were still some subtleties involved. First, why should the mass of the Earth act as if it were all placed at the center? Newton showed that for a perfect sphere with uniform mass distribution, all the mass may be assumed to be located at the center. (This calculation is difficult and can be found in Appendix 9A to this

chapter.) We assume for the present calculation that the Earth and the Moon are perfect spheres with uniform mass distribution.

Second, does this gravitational force between the Earth and the Moon form an action-reaction Third Law pair? When Newton first explained the Moon's motion in 1666, he had still not formulated the Third Law, which accounted for the long delay in the publication of the *Principia*. The link between the concept of force and the concept of an action-reaction pair of forces was the last piece needed to solve the puzzle of the effect of gravity on planetary orbits. Once Newton realized that the gravitational force between any two bodies forms an action-reaction pair, and satisfies both the Universal Law of Gravitation and his newly formulated Third Law, he was able to solve the oldest and most important physics problem of his time, the motion of the planets.

The test for the Universal Law of Gravitation was performed through experimental observation of the motion of planets, which turned out to be resoundingly successful. For almost 200 years, Newton's Universal Law was in excellent agreement with observation. A sign of more complicated physics ahead, the first discrepancy only occurred when a slight deviation of the motion of Mercury was experimentally confirmed in 1882. The prediction of this deviation was the first success of Einstein's Theory of General Relativity (formulated in 1915).

We can apply this Universal Law of Gravitation to calculate the period of the Moon's orbit around the Earth. The mass of the Moon is $m_1 = 7.36 \times 10^{22}$ kg and the mass of the Earth is $m_2 = 5.98 \times 10^{24}$ kg. The distance from the Earth to the Moon is $R_{e,m} = 3.82 \times 10^8$ m. We show the force diagram in Figure 9.2.
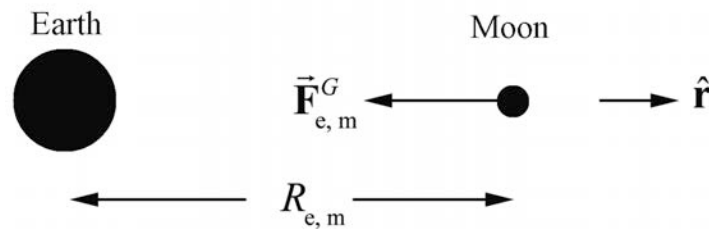

Figure 9.2 Gravitational force of moon

Newton's Second Law of motion for the radial direction becomes

$$-G\frac{m_1 m_2}{R_{e,m}^2} = -m_1\frac{4\pi^2 R_{e,m}}{T^2} .$$ (9.2.3)

We can solve this equation for the period of the orbit,

$$T = \sqrt{\frac{4\pi^2 R_{e,m}^3}{G m_2}} . \qquad (9.2.4)$$

Substitute the given values for the radius of the orbit, the mass of the earth, and the universal gravitational constant. The period of the orbit is

$$T = \sqrt{\frac{4\pi^2 (3.82 \times 10^8 \text{ m})^3}{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2})(5.98 \times 10^{24} \text{ kg})}} = 2.35 \times 10^6 \text{ s} . \qquad (9.2.5)$$

This period is given in days by

$$T = (2.35 \times 10^6 \text{ s}) \left(\frac{1 \text{ day}}{8.64 \times 10^4 \text{ s}} \right) = 27.2 \text{ days}. \qquad (9.2.6)$$

This period is called the *sidereal month* because it is the time that it takes for the Moon to return to a given position with respect to the stars.

The actual time T_1 between full moons, called the *synodic month* (the average period of the Moon's revolution with respect to the earth and is 29.53 days, it may range between 29.27 days and 29.83 days), is longer than the sidereal month because the Earth is traveling around the Sun. So for the next full moon, the Moon must travel a little farther than one full circle around the Earth in order to be on the other side of the Earth from the Sun (Figure 9.3).
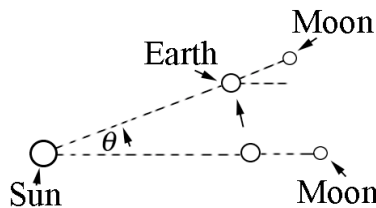

Figure 9.3: Orbital motion between full moons

Therefore the time T_1 between consecutive full moons is approximately $T_1 \simeq T + \Delta T$ where $\Delta T \simeq T / 12 = 2.3 \text{ days}$. So $T_1 \simeq 29.5 \text{ days}$.

9.2.2 Kepler's Third Law and Circular Motion

The first thing that we notice from the above solution is that the period does not depend on the mass of the Moon. We also notice that the square of the period is proportional to the cube of the distance between the Earth and the Moon,

$$T^2 = \frac{4\pi^2 R_{e,m}^3}{G m_2}.$$
(9.2.7)

This is an example of Kepler's Third Law, of which Newton was aware. This confirmation was convincing evidence to Newton that his Universal Law of Gravitation was the correct mathematical description of the gravitational force law, even though he still could not explain what "caused" gravity.

9.3 Worked Examples Circular Motion

Example 9.1 Geosynchronous Orbit

A geostationary satellite goes around the earth once every 23 hours 56 minutes and 4 seconds, (a sidereal day, shorter than the noon-to-noon solar day of 24 hours) so that its position appears stationary with respect to a ground station. The mass of the earth is $m_e = 5.98 \times 10^{24}$ kg. The mean radius of the earth is $R_e = 6.37 \times 10^6$ m. The universal constant of gravitation is $G = 6.67 \times 10^{-11}$ N $\cdot$ m^2 $\cdot$ kg^{-2}. What is the radius of the orbit of a geostationary satellite? Approximately how many earth radii is this distance?

Solution: The satellite's motion can be modeled as uniform circular motion. The gravitational force between the earth and the satellite keeps the satellite moving in a circle (In Figure 9.4, the orbit is close to a scale drawing of the orbit). The acceleration of the satellite is directed towards the center of the circle, that is, along the radially inward direction.
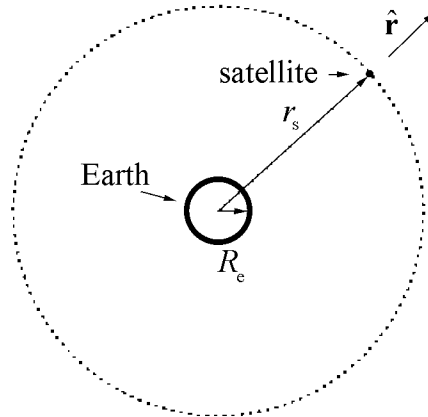

Figure 9.4 Geostationary satellite orbit (close to a scale drawing of orbit).

Choose the origin at the center of the earth, and the unit vector $\hat{\mathbf{r}}$ along the radial direction. This choice of coordinates makes sense in this problem since the direction of acceleration is along the radial direction.

Let $\vec{r}$ be the position vector of the satellite. The magnitude of $\vec{r}$ (we denote it as r_s) is the distance of the satellite from the center of the earth, and hence the radius of its circular orbit. Let ω be the angular velocity of the satellite, and the period is $T = 2\pi / \omega$. The acceleration is directed inward, with magnitude $r_s\omega^2$; in vector form,

$$\vec{a} = -r_s\omega^2\hat{r}. \tag{9.3.1}$$

Apply Newton's Second Law to the satellite for the radial component. The only force in this direction is the gravitational force due to the Earth,

$$\vec{F}_{grav} = -m_s\omega^2 r_s\,\hat{r}. \tag{9.3.2}$$

The inward radial force on the satellite is the gravitational attraction of the earth,

$$-G\frac{m_s m_e}{r_s^2}\hat{r} = -m_s\omega^2 r_s\,\hat{r}. \tag{9.3.3}$$

Equating the $\hat{r}$ components,

$$G\frac{m_s m_e}{r_s^2} = m_s\omega^2 r_s. \tag{9.3.4}$$

Solving for the radius of orbit of the satellite r_s,

$$r_s = \left(\frac{G m_e}{\omega^2}\right)^{1/3}. \tag{9.3.5}$$

The period T of the satellite's orbit in seconds is 86164 s and so the angular speed is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{86164\text{ s}} = 7.2921\times10^{-5}\,\text{s}^{-1}. \tag{9.3.6}$$

Using the values of ω, G and m_e in Equation (9.3.5), we determine r_s,

$$r_s = 4.22\times10^7\,\text{m} = 6.62\,R_e. \tag{9.3.7}$$

Example 9.2 Double Star System

Consider a double star system under the influence of gravitational force between the stars. Star 1 has mass m_1 and star 2 has mass m_2. Assume that each star undergoes uniform circular motion such that the stars are always a fixed distance s apart (rotating counterclockwise in Figure 9.5). What is the period of the orbit?
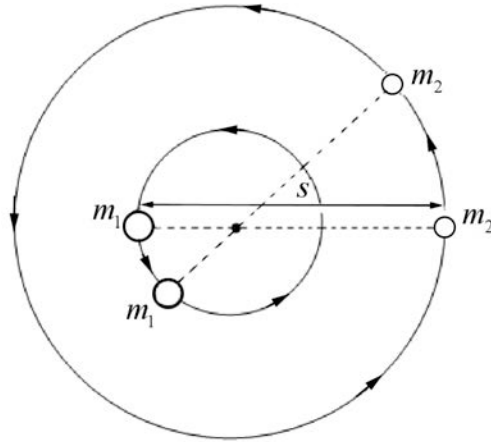
Figure 9.5 Two stars undergoing circular orbits about each other

Solution: Because the distance between the two stars doesn't change as they orbit about each other, there is a central point where the lines connecting the two objects intersect as the objects move, as can be seen in the figure above. (We will see later on in the course that central point is the center of mass of the system.) Choose radial coordinates for each star with origin at that central point. Let $\hat{\mathbf{r}}_1$ be a unit vector at Star 1 pointing radially away from the center of mass. The position of object 1 is then $\vec{\mathbf{r}}_1 = r_1\,\hat{\mathbf{r}}_1$, where r_1 is the distance from the central point. Let $\hat{\mathbf{r}}_2$ be a unit vector at Star 2 pointing radially away from the center of mass. The position of object 2 is then $\vec{\mathbf{r}}_2 = r_2\,\hat{\mathbf{r}}_2$, where r_2 is the distance from the central point. Because the distance between the two stars is fixed we have that

$$s = r_1 + r_2 \,.$$

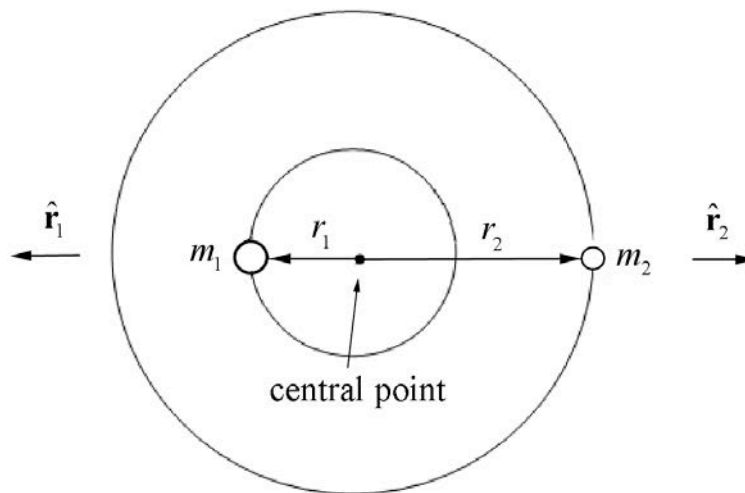The coordinate system is shown in Figure 9.6

Figure 9.6 Coordinate system for double star orbits

The gravitational force on object 1 is then

$$\vec{F}_{2,1} = -\frac{Gm_1m_2}{s^2}\hat{r}_1 .$$

The gravitational force on object 2 is then

$$\vec{F}_{1,2} = -\frac{Gm_1m_2}{s^2}\hat{r}_2 .$$

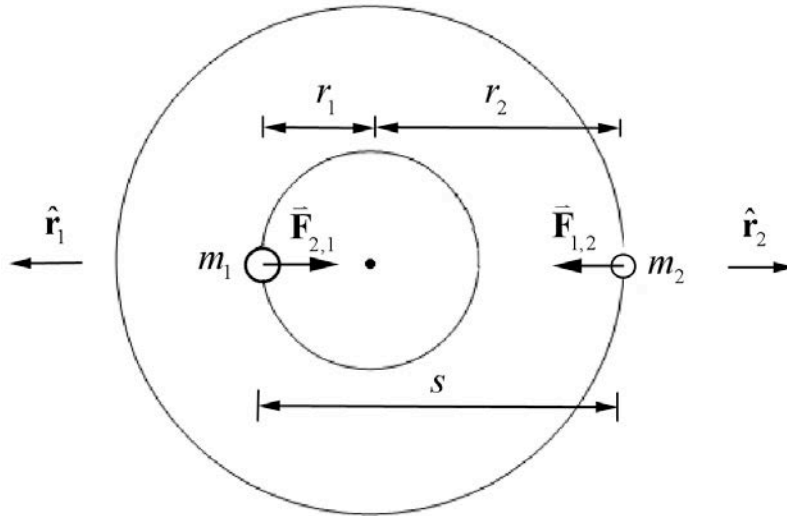The force diagrams on the two stars are shown in Figure 9.7.

Figure 9.7 Force diagrams on objects 1 and 2

Let ω denote the magnitude of the angular velocity of each star about the central point. Then Newton's Second Law, $\vec{F}_1 = m_1\vec{a}_1$, for Star 1 in the radial direction $\hat{r}_1$ is

$$-G\frac{m_1 m_2}{s^2} = -m_1 r_1 \omega^2 .$$

We can solve this for r_1,

$$r_1 = G\frac{m_2}{\omega^2 s^2} .$$

Newton's Second Law, $\vec{F}_2 = m_2\vec{a}_2$, for Star 2 in the radial direction $\hat{r}_2$ is

$$-G\frac{m_1 m_2}{s^2} = -m_2 r_2 \omega^2 .$$

We can solve this for r_2,

$$r_2 = G\frac{m_1}{\omega^2 s^2} .$$

Because s, the distance between the stars, is constant

$$s = r_1 + r_2 = G\frac{m_2}{\omega^2 s^2} + G\frac{m_1}{\omega^2 s^2} = G\frac{(m_2 + m_1)}{\omega^2 s^2} .$$

Thus the magnitude of the angular velocity is

$$\omega = \left(G\frac{(m_2 + m_1)}{s^3} \right)^{1/2} ,$$

and the period is then

$$T = \frac{2\pi}{\omega} = \left(\frac{4\pi^2 s^3}{G(m_2 + m_1)} \right)^{1/2} . \tag{9.3.8}$$

Note that both masses appear in the above expression for the period unlike the expression for Kepler's Law for circular orbits. Eq. (9.2.7). The reason is that in the argument leading up to Eq. (9.2.7), we assumed that $m_1 \ll m_2$, this was equivalent to assuming that the central point was located at the center of the Earth. If we used Eq. (9.3.8) instead we would find that the orbital period for the circular motion of the Earth and moon about each other is

$$T = \sqrt{\frac{4\pi^2 (3.82 \times 10^8 \text{ m})^3}{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2})(5.98 \times 10^{24} \text{ kg} + 7.36 \times 10^{22} \text{ kg})}} = 2.33 \times 10^6 \text{ s} ,$$

which is $1.43 \times 10^4 \text{ s} = 0.17 \text{ d}$ shorter than our previous calculation.

Example 9.3 Rotating Objects

Two objects 1 and 2 of mass m_1 and m_2 are whirling around a shaft with a constant angular velocity ω. The first object is a distance d from the central axis, and the second object is a distance $2d$ from the axis (Figure 9.8). You may ignore the mass of the strings and neglect the effect of gravity. (a) What is the tension in the string between the inner

object and the outer object? (b) What is the tension in the string between the shaft and the inner object?
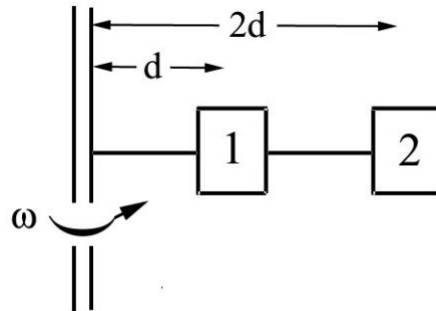

Figure 9.8 Objects attached to a rotating shaft

Solution: We begin by drawing separate force diagrams, Figure 9.9a for object 1 and Figure 9.9b for object 2.

(a)   (b)
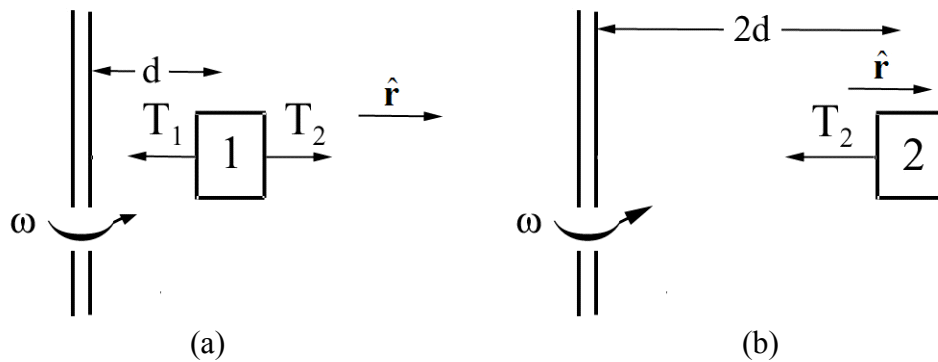
Figure 9.9 (a) and 9.9 (b) Free-body force diagrams for objects 1 and 2

Newton's Second Law, $\vec{F}_1 = m_1 \vec{a}_1$, for the inner object in the radial direction is

$$\hat{r} : T_2 - T_1 = -m_1 d \omega^2 .$$

Newton's Second Law, $\vec{F}_2 = m_2 \vec{a}_2$, for the outer object in the radial direction is

$$\hat{r} : - T_2 = -m_2 2d \omega^2 .$$

The tension in the string between the inner object and the outer object is therefore

$$T_2 = m_2 2d \omega^2 .$$

Using this result for T_2 in the force equation for the inner object yields

$$m_2 2d\,\omega^2 - T_1 = -m_1 d\,\omega^2,$$

which can be solved for the tension in the string between the shaft and the inner object

$$T_1 = d\,\omega^2(m_1 + 2m_2).$$

Example 9.4 Tension in a Rope

A uniform rope of mass m and length L is attached to shaft that is rotating at constant angular velocity ω. Find the tension in the rope as a function of distance from the shaft. You may ignore the effect of gravitation.

Solution: Divide the rope into small pieces of length Δr, each of mass $\Delta m = (m\,/\,L)\Delta r$. Consider the piece located a distance r from the shaft (Figure 9.10).
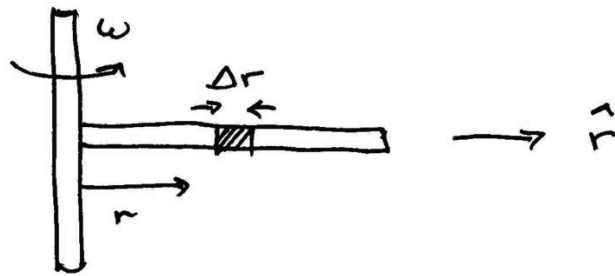

Figure 9.10 Small slice of rotating rope

The radial component of the force on that piece is the difference between the tensions evaluated at the sides of the piece, $F_r = T(r + \Delta r) - T(r)$, (Figure 9.11).
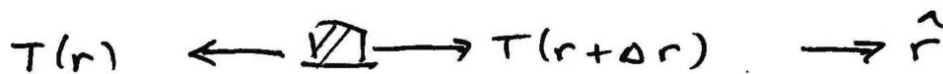

Figure 9.11 Free-body force diagram on small slice of rope

The piece is accelerating inward with a radial component $a_r = -r\,\omega^2$. Thus Newton's Second Law becomes

$$F_r = -\Delta m\,\omega^2 r$$

$$T(r + \Delta r) - T(r) = -(m\,/\,L)\Delta r\,r\omega^2.$$

(9.3.9)

Denote the difference in the tension by $\Delta T = T(r + \Delta r) - T(r)$. After dividing through by Δr, Eq. (9.3.9) becomes

$$\frac{\Delta T}{\Delta r} = -(m / L)\, r\omega^2 . \tag{9.3.10}$$

In the limit as $\Delta r \to 0$, Eq. (9.3.10) becomes a differential equation,

$$\frac{dT}{dr} = -(m / L)\omega^2 r . \tag{9.3.11}$$

From this, we see immediately that the tension decreases with increasing radius. We shall solve this equation by integration

$$
\begin{aligned}
T(r) - T(L) &= \int_{r'=L}^{r'=r} \frac{dT}{dr'}\, dr' = -(m\omega^2 / L)\int_{L}^{r} r'\, dr' \\
&= -(m\omega^2 / 2L)(r^2 - L^2) \\
&= (m\omega^2 / 2L)(L^2 - r^2).
\end{aligned} \tag{9.3.12}
$$

We use the fact that the tension, in the ideal case, will vanish at the end of the rope, $r = L$. Thus,

$$T(r) = (m\omega^2 / 2L)(L^2 - r^2). \tag{9.3.13}$$

This last expression shows the expected functional form, in that the tension is largest closest to the shaft, and vanishes at the end of the rope.

Example 9.5 Object Sliding in a Circular Orbit on the Inside of a Cone

Consider an object of mass m that slides without friction on the inside of a cone moving in a circular orbit with constant speed v_0. The cone makes an angle θ with respect to a vertical axis. The axis of the cone is vertical and gravity is directed downwards. The apex half-angle of the cone is θ as shown in Figure 9.12. Find the radius of the circular path and the time it takes to complete one circular orbit in terms of the given quantities and g.
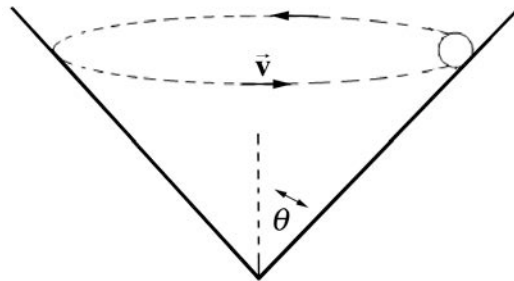

Figure 9.12 Object in a circular orbit on inside of a cone

Solution: Choose cylindrical coordinates as shown in the above figure. Choose unit vectors $\hat{\mathbf{r}}$ pointing in the radial outward direction and $\hat{\mathbf{k}}$ pointing upwards. The force diagram on the object is shown in Figure 9.13.
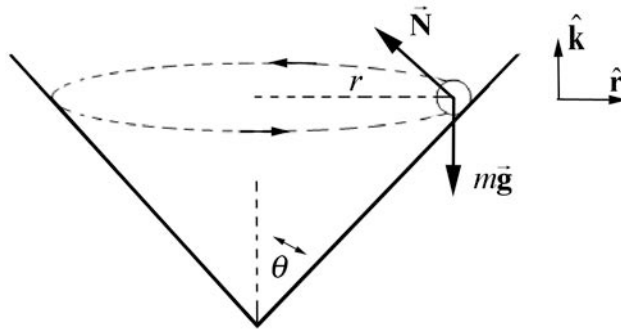

Figure 9.13 Free-body force diagram on object

The two forces acting on the object are the normal force of the wall on the object and the gravitational force. Then Newton's Second Law in the $\hat{\mathbf{r}}$-direction becomes

$$-N\cos\theta = \frac{-mv^2}{r}$$

and in the $\hat{\mathbf{k}}$-direction becomes

$$N\sin\theta - mg = 0.$$

These equations can be re-expressed as

$$N\cos\theta = m\frac{v^2}{r}$$

$$N\sin\theta = mg.$$

We can divide these two equations,

$$\frac{N\sin\theta}{N\cos\theta} = \frac{mg}{mv^2/r}$$

yielding

$$\tan\theta = \frac{rg}{v^2}.$$

This can be solved for the radius,

$$r = \frac{v^2}{g}\tan\theta.$$

The centripetal force in this problem is the vector component of the contact force that is pointing radially inwards,

$$F_{\text{cent}} = N\cos\theta = mg\cot\theta,$$

where $N\sin\theta = mg$ has been used to eliminate N in terms of m, g and θ. The radius is independent of the mass because the component of the normal force in the vertical direction must balance the gravitational force, and so the normal force is proportional to the mass.

Example 9.6 Coin on a Rotating Turntable

A coin of mass m (which you may treat as a point object) lies on a turntable, exactly at the rim, a distance R from the center. The turntable turns at constant angular speed ω and the coin rides without slipping. Suppose the coefficient of static friction between the turntable and the coin is given by μ. Let g be the gravitational constant. What is the maximum angular speed ω_{max} such that the coin does not slip?
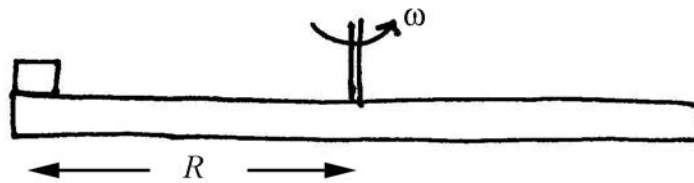

Figure 9.14 Coin on Rotating Turntable

Solution: The coin undergoes circular motion at constant speed so it is accelerating inward. The force inward is static friction and at the just slipping point it has reached its maximum value. We can use Newton's Second Law to find the maximum angular speed ω_{max}. We choose a polar coordinate system and the free-body force diagram is shown in the figure below.
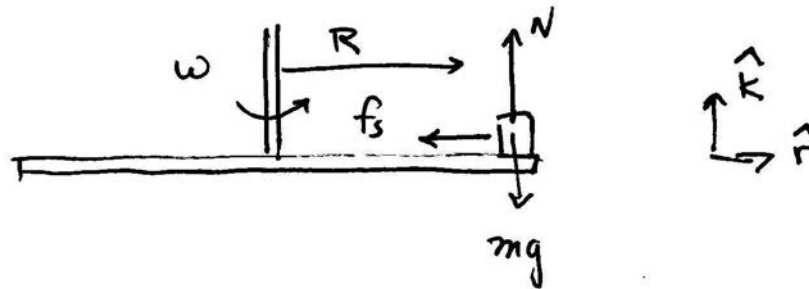

Figure 9.15 Free-body force diagram on coin

The contact force is given by
$$\vec{C} = \vec{N} + \vec{f}_s = N\hat{k} - f_s\hat{r}.$$
(9.3.14)

The gravitational force is given by
$$\vec{F}_{grav} = -mg\hat{k}.$$
(9.3.15)

Newton's Second Law in the radial direction is given by

$$-f_{\mathrm{s}} = -m R \omega^2 . \tag{9.3.16}$$

Newton's Second Law, $F_z = ma_z$, in the z-direction, noting that the disc is static hence $a_z = 0$, is given by

$$N - mg = 0 . \tag{9.3.17}$$

Thus the normal force is

$$N = mg . \tag{9.3.18}$$

As ω increases, the static friction increases in magnitude until at $\omega = \omega_{\mathrm{max}}$ and static friction reaches its maximum value (noting Eq. (9.3.18)).

$$(f_{\mathrm{s}})_{\mathrm{max}} = \mu N = \mu mg . \tag{9.3.19}$$

At this value the disc slips. Thus substituting this value for the maximum static friction into Eq. (9.3.16) yields

$$\mu mg = mR\omega_{\mathrm{max}}^2 . \tag{9.3.20}$$

We can now solve Eq. (9.3.20) for maximum angular speed ω_{max} such that the coin does not slip

$$\omega_{\mathrm{max}} = \sqrt{\frac{\mu g}{R}} . \tag{9.3.21}$$

Appendix 9A The Gravitational Field of a Spherical Shell of Matter

When analyzing gravitational interactions between uniform spherical bodies we assumed we could treat each sphere as a point-like mass located at the center of the sphere and then use the Universal Law of Gravitation to determine the force between the two point-like objects. We shall now justify that assumption. For simplicity we only need to consider the interaction between a spherical object and a point-like mass. We would like to determine the gravitational force on the point-like object of mass m_1 due to the gravitational interaction with a solid uniform sphere of mass m_2 and radius R. In order to determine the force law we shall first consider the interaction between the point-like object and a uniform spherical shell of mass m_s and radius R. We will show that:

1) The gravitational force acting on a point-like object of mass m_1 located a distance $r > R$ from the center of a uniform spherical shell of mass m_s and radius R is the same force that would arise if all the mass of the shell were placed at the center of the shell.

2) The gravitational force on an object of mass m_1 placed inside a spherical shell of matter is zero.

The force law summarizes these results:

$$\vec{\mathbf{F}}_{s,1}(r) = \begin{cases} -G\dfrac{m_s m_1}{r^2}\hat{\mathbf{r}}, & r > R \\ \vec{\mathbf{0}}, & r < R \end{cases},$$

where $\hat{\mathbf{r}}$ is the unit vector located at the position of the object and pointing radially away from the center of the shell.

For a uniform spherical distribution of matter, we can divide the sphere into thin shells. Then the force between the point-like object and each shell is the same as if all the mass of the shell were placed at the center of the shell. Then we add up all the contributions of the shells (integration), the spherical distribution can be treated as point-like object located at the center of the sphere justifying our assumption.

Thus it suffices to analyze the case of the spherical shell. We shall first divide the shell into small area elements and calculate the gravitational force on the point-like object due to one element of the shell and then add the forces due to all these elements via integration.

We begin by choosing a coordinate system. Choose our z-axis to be directed from the center of the sphere to the position of the object, at position $\vec{r} = z\hat{k}$, so that $z \geq 0$. (Figure 9A.1 shows the object lying outside the shell with $z > R$).
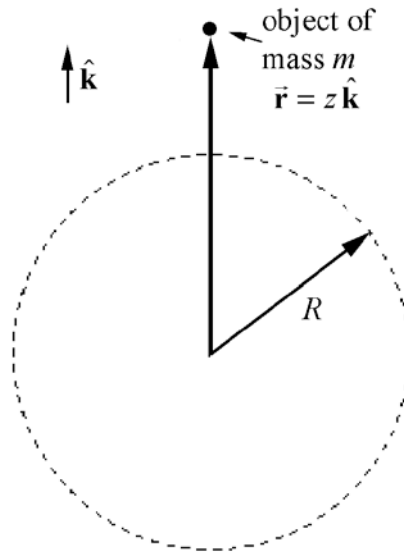

Figure 9A.1 Object lying outside shell with $z > R$.

Choose spherical coordinates as shown in Figure 9A.2.
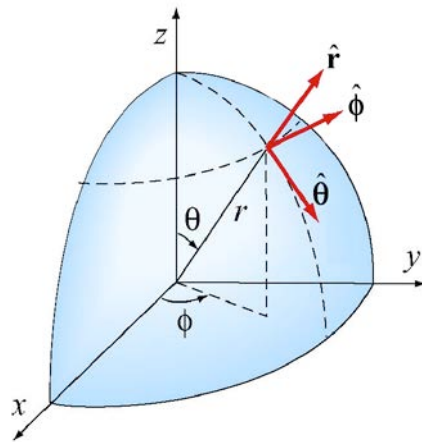

Figure 9A.2 Spherical coordinates

For a point on the surface of a sphere of radius $r = R$, the Cartesian coordinates are related to the spherical coordinates by

$$
\begin{aligned}
x &= R\sin\theta\cos\phi, \\
y &= R\sin\theta\sin\phi, \\
z &= R\cos\theta,
\end{aligned}
\tag{9.A.1}
$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

Note that the angle θ in Figure 9A.2 and Equations (9.A.1) is not the same as that in plane polar coordinates or cylindrical coordinates. The angle θ is known as the *co-latitude*, the complement of the latitude. We now choose a small area element shown in Figure 9A.3.
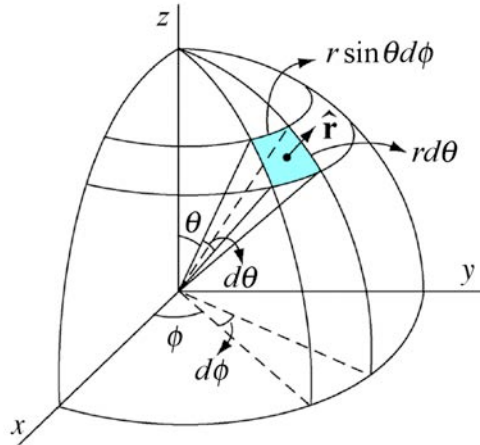

Figure 9A.3 Infinitesimal area element

The infinitesimal area element on the surface of the shell is given by

$$da = R^2 \sin\theta d\theta d\phi .$$

Then the mass dm contained in that element is

$$dm = \sigma da = \sigma R^2 \sin\theta d\theta d\phi .$$

where σ is the surface mass density given by

$$\sigma = m_s / 4\pi R^2 .$$

The gravitational force $\vec{\mathbf{F}}_{dm, m_1}$ on the object of mass m_1 that lies outside the shell due to the infinitesimal piece of the shell (with mass dm) is shown in Figure 9A.4.
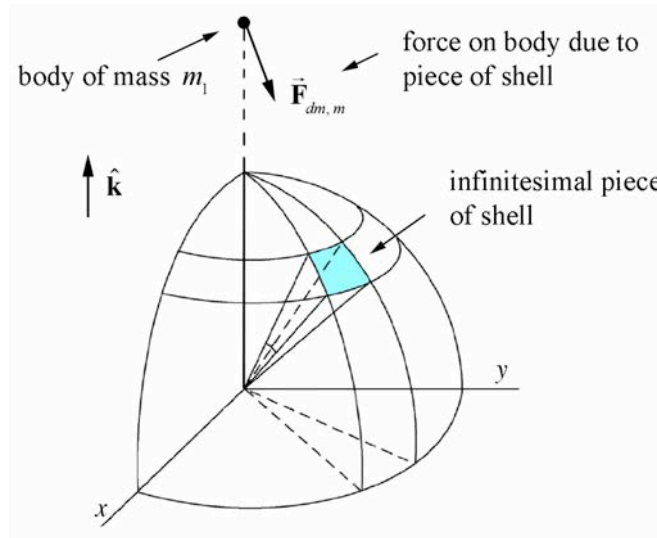
Figure 9A.4 Force on a point-like object due to piece of shell

The contribution from the piece with mass dm to the gravitational force on the object of mass m_1 that lies outside the shell has a component pointing in the negative $\hat{\mathbf{k}}$ -direction and a component pointing radially away from the z -axis. By symmetry there is another mass element with the same differential mass $dm' = dm$ on the other side of the shell with same co-latitude θ but with ϕ replaced by $\phi \pm \pi$; this replacement changes the sign of x and y in Equations (9.A.1) but leaves z unchanged. This other mass element produces a gravitational force that exactly cancels the radial component of the force pointing away from the z -axis. Therefore the sum of the forces of these differential mass elements on the object has only a component in the negative $\hat{\mathbf{k}}$ -direction (Figure 9A.5)
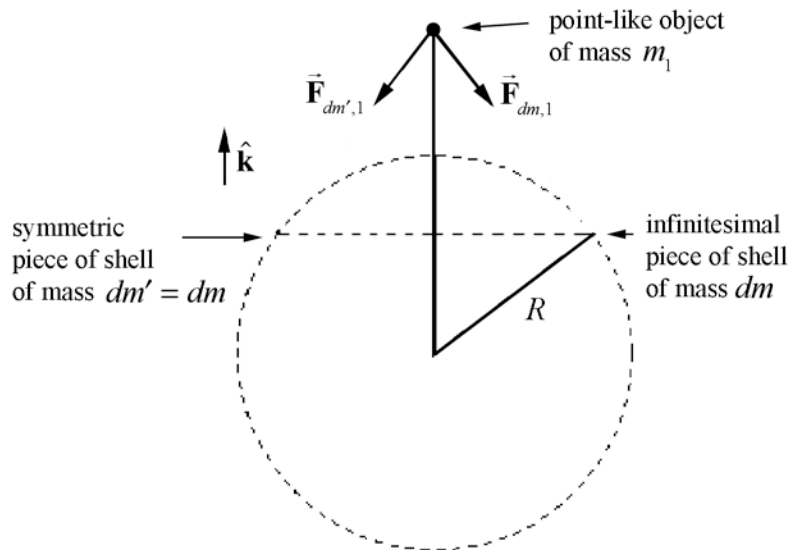

Figure 9A.5 Symmetric cancellation of components of force

Therefore we need only the z-component vector of the force due to the piece of the shell on the point-like object.
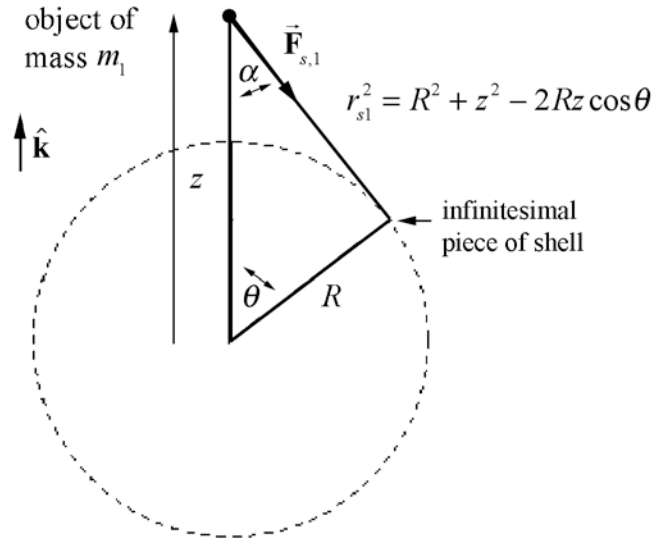

Figure 9A.6 Geometry for calculating the force due to piece of shell.

From the geometry of the set-up (Figure 9A.6) we see that

$$(d\vec{F}_{s,1})_z \equiv dF_z \hat{k} = -G \frac{m_1 dm}{r_{s1}^2} \cos\alpha \hat{k}.$$

Thus

$$dF_z = -G \frac{m_1 dm}{r_{s1}^2} \cos\alpha = -\frac{Gm_s m_1}{4\pi} \frac{\cos\alpha \sin\theta d\theta d\phi}{r_{s1}^2}. \qquad (9.A.2)$$

The integral of the force over the surface is then

$$F_z = -Gm_1 \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \frac{dm \cos\alpha}{r_{s1}^2} = -\frac{Gm_s m_1}{4\pi} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \frac{\cos\alpha \sin\theta d\theta d\phi}{r_{s1}^2}. \qquad (9.A.3)$$

The ϕ-integral is straightforward yielding

$$F_z = -\frac{Gm_s m_1}{2} \int_{\theta=0}^{\theta=\pi} \frac{\cos\alpha \sin\theta d\theta}{r_{s1}^2}. \qquad (9.A.4)$$

From Figure 9A.6 we can use the law of cosines in two different ways

$$r_{s1}^2 = R^2 + z^2 - 2 R z \cos \theta$$

$$R^2 = z^2 + r_{s,1}^2 - 2 r_{s,1} z \cos \alpha. \tag{9.A.5}$$

Differentiating the first expression in (9.A.5), with R and z constant yields,

$$2 r_{s,1} \, dr_{s,1} = 2 R z \sin \theta \, d\theta. \tag{9.A.6}$$

Hence

$$\sin \theta \, d\theta = \frac{r_{s,1}}{R z} dr_{s,1}. \tag{9.A.7}$$

and from the second expression in (9.A.5) we have that

$$\cos \alpha = \frac{1}{2 z r_{s,1}} \left[(z^2 - R^2) + r_{s1}^2 \right]. \tag{9.A.8}$$

We now have everything we need in terms of $r_{s,1}$.

For the case when $z > R$, $r_{s,1}$ varies from $z - R$ to $z + R$. Substituting Equations (9.A.7) and (9.A.8) into Eq. (9.A.3) and using the limits for the definite integral yields

$$
\begin{aligned}
F_z &= -\frac{Gm_s m_1}{2} \int_{\theta=0}^{\theta=\pi} \frac{\cos \alpha \sin \theta}{r_{s,1}^2} d\theta \\
&= -\frac{Gm_s m_1}{2} \frac{1}{2z} \int_{z-R}^{z+R} \frac{1}{r_{s,1}} \left[(z^2 - R^2) + r_{s,1}^2 \right] \frac{1}{r_{s,1}^2} \frac{r_{s,1} \, dr_{s,1}}{R z} \\
&= -\frac{Gm_s m_1}{2} \frac{1}{2 R z^2} \left[(z^2 - R^2) \int_{z-R}^{z+R} \frac{dr_{s,1}}{r_{s,1}^2} + \int_{z-R}^{z+R} dr_{s,1} \right].
\end{aligned} \tag{9.A.9}
$$

No tables should be needed for these; the result is

$$
\begin{aligned}
F_z &= -\frac{Gm_s m_1}{2} \frac{1}{2 R z^2} \left[-\frac{(z^2 - R^2)}{r_{s,1}} + r_{s,1} \right]_{z-R}^{z+R} \\
&= -\frac{Gm_s m_1}{2} \frac{1}{2 R z^2} \left[-(z - R) + (z + R) + 2 R \right] \\
&= -\frac{Gm_s m_1}{z^2}.
\end{aligned} \tag{9.A.10}
$$

For the case when $z < R$, $r_{s,1}$ varies from $R - z$ to $R + z$. Then the integral is

$$F_z = -\frac{Gm_sm_1}{2}\frac{1}{2Rz^2}\left[-\frac{\left(z^2 - R^2\right)}{r_{s,1}} + r_{s,1}\right]_{R-z}^{R+z}$$

$$= -\frac{Gm_sm_1}{2}\frac{1}{2Rz^2}\left[-\left(z - R\right)-\left(z + R\right)+2z\right] \qquad (9.\text{A}.11)$$

$$= 0.$$

So we have demonstrated the proposition that for a point-like object located on the z-axis a distance z from the center of a spherical shell, the gravitational force on the point like object is given by

$$\vec{\mathbf{F}}_{s,1}(r) = \begin{cases} -G\dfrac{m_sm_1}{z^2}\hat{\mathbf{k}}, & z > R \\ \vec{\mathbf{0}}, & z < R \end{cases}.$$

This proves the result that the gravitational force inside the shell is zero and the gravitational force outside the shell is equivalent to putting all the mass at the center of the shell.

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