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Supplementary notes:
Complex numbers

A.1 Introduction & Rant

Complex numbers are numbers that consist of a real and an imaginary part. As we will hopefully come to realize very soon, complex numbers are one of the most fantastically useful mathematical concepts that have been devised.

Sadly, they are encumbered by some rather unfortunate terminology. In particular, *there is nothing imaginary about imaginary numbers!!!* We (hopefully) all know that the imaginary number i is defined by

$$i^2 = -1$$

i.e.,

$$i = \sqrt{-1}\ .$$

People are often taught that this is an "imaginary" number because we can't visualize or conceptualize what the square root of a negative number "really" is. This is absolute garbage. The only thing that is "imaginary" about "imaginary numbers" is that it took an act of imagination to conceive of them in the first place. This is hardly unprecedented: it took an act of imagination to conceive of negative numbers ("How can someone have negative six rocks?"), or irrational numbers. In a manner similar to negative numbers or irrational numbers, imaginary numbers weren't invented in some fit of self-perpetuating mathematical foolishness — they are a useful, meaningful concept that allow us to greatly extend the scope of problems that can be solved mathematically. As a particular example that we will soon see demonstrates, they make solving second order differential equations a breeze.

End rant.

A.2 Complex numbers: definitions and representation

As mentioned above, complex numbers are numbers with both a real and an imaginary part. They thus have the form

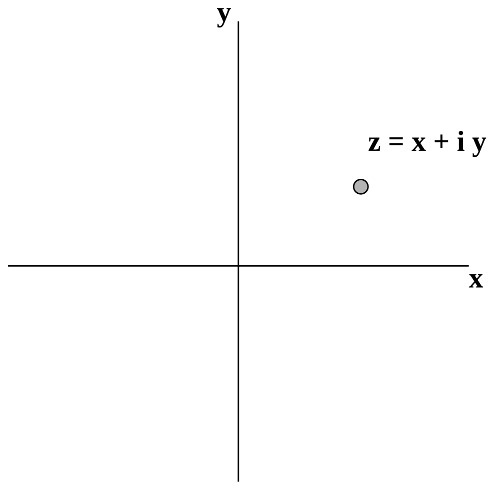
$$z = x + iy$$

where, of course, $i = \sqrt{-1}$. A useful auxiliary concept is the "complex conjugate" of a complex number: the complex conjugate of a complex number z is given by

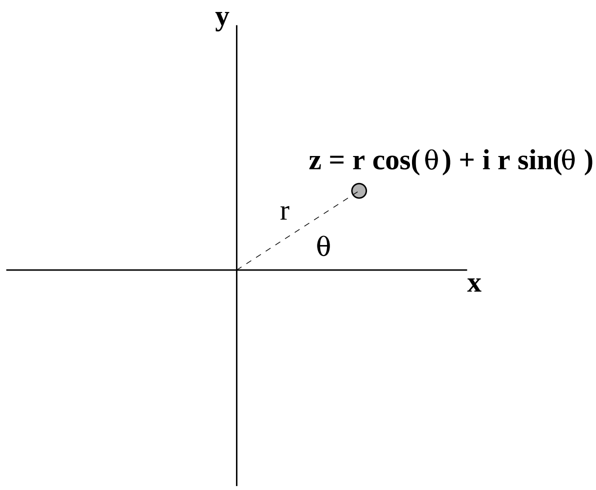
$$z^* = x - iy\ .$$

In other words, it has the same real part but the imaginary part has the opposite sign. We'll come back to the complex conjugate shortly.

This way of writing complex numbers suggests a nice way to represent them: we think of $z = x + iy$ as a coordinate in the (x, y) plane:



This is called the "complex plane" representation. Any complex number is given by a point in this plane. Using simple trigonometry, this point can be written in terms of a "magnitude" $|z| = r = \sqrt{x^2 + y^2}$ and an angle or "phase" $\theta = \arctan(y/x)$:



In what follows, we will find this second representation to be particularly useful.

A.3 Euler's relation

One of the main reasons that complex numbers are so useful in physics is the following amazing relationship:

$$e^{i\theta} = \cos\theta + i\sin\theta\ .$$

A simple way to prove this is to expand the exponential in a Maclaurin series:

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \frac{1}{2}(i\theta)^2 + \frac{1}{6}(i\theta)^3 + \dots \\ &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}\ . \end{aligned}$$

Now, we reorganize this formula. To do so, note the following powers of i:

$$\begin{aligned} i^0 &= 1 \\ i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \\ i^5 &= i \qquad \text{etc.} \end{aligned}$$

Then,

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty}(-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \\ &= \cos\theta + i \sin\theta \ . \end{aligned}$$

The second line follows from carefully reorganizing the Maclaurin expansion of the exponential, recognizing that the odd power terms contain an overall factor of i, the even terms do not. The third line follows from recognizing that the two sums following this reorganization are just the Maclaurin expansions of the sine and cosine.

Notice that the complex conjugate of z is likewise simple:

$$\begin{aligned} z^* &= x - iy \\ &= re^{-i\theta} \ . \end{aligned}$$

A.4 Using the Euler relation

The Euler relation is fantastic. First, it tells us that the polar coordinate representation of a complex number is *very* simple:

$$\begin{aligned} z &= x + iy \\ &= r\cos\theta + ir\sin\theta \\ &= re^{i\theta} \ . \end{aligned}$$

This leads to the "phasor" representation of complex numbers: essentially, the complex number is represented as a displacement vector from the origin — the "phasor" — to the point (x, y) in the plane. The length of this phasor is r; θ is the angle from the x axis to the phasor.

Let's look at a couple of particularly important complex numbers. In all cases, we put $r = 1$. Imagine we increase θ from 0 — the phasor on the x-axis — to 2π, in steps of $\pi/2$:

$$\begin{aligned} \theta = 0 &\quad\longrightarrow\quad z = 1 \\ \theta = \frac{\pi}{2} &\quad\longrightarrow\quad z = e^{i\pi/2} = i \\ \theta = \pi &\quad\longrightarrow\quad z = e^{i\pi} = -1 \\ \theta = \frac{3\pi}{2} &\quad\longrightarrow\quad z = e^{3i\pi/2} = -i \\ \theta = 2\pi &\quad\longrightarrow\quad z = e^{2i\pi} = 1 \ . \end{aligned}$$

An interesting and important phasor is one in which the angle grows with time:

$$z(t) = re^{i\omega t} \ .$$

Assume for now that ω is a purely real number. In the complex plane, this phasor looks like a vector of length r sweeping around counterclockwise at constant angular velocity ω; the phasor traces out a circle in the complex plane. Its complex conjugate,

$$z^*(t) = re^{-i\omega t} \ ,$$

looks like a vector of length r sweeping around clockwise with constant angular velocity ω.

What if ω is itself complex? Suppose we have

$$\omega = \omega_r + i\omega_i \ .$$

Then,

$$\begin{aligned} z(t) &= re^{i\omega t} = re^{i\omega_r t - \omega_i t} \\ &= re^{-\omega_i t}e^{i\omega_r t} \ . \end{aligned}$$

This is again a vector sweeping around counterclockwise, with angular velocity ω_r. In this case, though, the length of the vector is exponentially decaying; the phasor traces out a spiral in the complex plane, asymptoting to the origin. For the complex conjugate, we have

$$\begin{aligned} z^*(t) &= re^{-i\omega^* t} = re^{-i\omega_r t - \omega_i t} \\ &= re^{-\omega_i t}e^{-i\omega_r t} \ . \end{aligned}$$

This is a decaying vector sweeping around clockwise.

A.5 Solving the harmonic oscillator

A harmonic oscillator is governed by the equation

$$ma = -kx$$

where a is acceleration. This provides us with the differential equation

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \ .$$

We know (hopefully) that the solution to this equation can be written as sines and cosines:

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

where $\omega_0 = \sqrt{k/m}$, and the constants A and B are chosen to match initial conditions. For example if $x(0) = x_0$ and $v(0) = (dx/dt)|_{t=0} = 0$, then

$$A = x_0, \qquad B = 0 \ .$$

An equivalent way to write this solution is to put

$$x = A\cos(\omega_0 t + \phi_0)$$

and match initial conditions by adjusting the constants A and ϕ_0. With the initial conditions chosen above, we would put $A = x_0$ and $\phi_0 = 0$.

There is yet a third way to write our solution: we put

$$x(t) = \mathrm{Re}\left[\tilde{x}(t)\right]$$

where Re$[z]$ means "the real part of z", and we put

$$\tilde{x}(t) = Ae^{i(\omega_0 t + \phi_0)} \ .$$

The reason that this method works so well is that $\tilde{x}(t)$ also solves the differential equation:

$$\frac{d^2\tilde{x}}{dt^2} + \frac{k}{m}\tilde{x} = 0$$

as long as we have $\omega_0 = \sqrt{k/m}$, exactly as before. This is easy to see:

$$\begin{aligned}
\frac{d\tilde{x}}{dt} &= i\omega_0 Ae^{i(\omega_0 t + \phi_0)} \\
&= i\omega_0 \tilde{x} \\
\frac{d^2\tilde{x}}{dt^2} &= -\omega_0^2 Ae^{i(\omega_0 t + \phi_0)} \\
&= -\omega_0^2 \tilde{x} \ .
\end{aligned}$$

Each time we take a derivative, we just pull down a factor of $i\omega_0$. Plugging into the differential equation shows us that $\omega_0 = \sqrt{k/m}$. We can then solve for initial conditions by choosing the constants A and ϕ_0. Once we have determined $\tilde{x}(t)$, we get $x(t)$ by just taking the real part.

Using complex numbers in this case is overkill. We now look at an example where it's more appropriate.

A.6 Solving the damped harmonic oscillator

The damped harmonic oscillator is governed by the differential equation

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \frac{k}{m}x = 0$$

where the constant γ tells us about drag forces that are proportional to the speed of the mass.

Solving this equation is kind of messy — as you hopefully learned in 8.012, you make some assumption about the form of the solution, plug that assumption in, grind through some hideous calculation, and end up with a damped sinusoidal oscillation.

All of that hideous calculation can be done with just a few lines of algebra using complex numbers. We proceed in exactly the same way as we did with the harmonic oscillator: we assume that $x(t) = \mathrm{Re}[\tilde{x}(t)]$, that $\tilde{x}(t) = Ae^{i(\alpha t + \phi_0)}$, and that $\tilde{x}(t)$ works in the damped oscillator equation. (I use α rather than ω_0 in the exponential because I don't want to assume *ab initio* that I will get some particular frequency — I want the math to show me what frequency is needed.) We need to use the facts that

$$\begin{aligned}
\frac{d\tilde{x}}{dt} &= i\alpha\, Ae^{i(\alpha t + \phi_0)} \\
&= i\alpha\, \tilde{x}(t)
\end{aligned}$$

and

$$\begin{aligned}\frac{d^2\tilde{x}}{dt^2} &= -\alpha^2\, Ae^{i(\alpha t+\phi_0)}\\ &= -\alpha^2\, \tilde{x}(t)\ .\end{aligned}$$

Each derivative brings down a power of $i\alpha$. When we plug these into the differential equation, we thus just pull out an overall factor of $\tilde{x}$:

$$\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{d\tilde{x}}{dt} + \frac{k}{m}\tilde{x} = 0$$

becomes

$$\tilde{x}(t)\left(-\alpha^2 + i\gamma\alpha + \frac{k}{m}\right) = 0$$

which means that

$$-\alpha^2 + im\alpha + \frac{k}{m} = 0\ .$$

This is a simple quadratic equation; its solution is

$$\alpha = i\frac{\gamma}{2} \pm \sqrt{\frac{k}{m} - \frac{\gamma^2}{4}}\ .$$

Notice what we get when we evaluate $e^{i\alpha t}$:

$$\begin{aligned}e^{i\alpha t} &= e^{-\gamma t/2} e^{\pm i\sqrt{\frac{k}{m}-\frac{\gamma^2}{4}}t}\\ &= e^{-\gamma t/2} e^{\pm i\omega t}\ .\end{aligned}$$

To keep the notation simple, I've defined

$$\omega \equiv \sqrt{\frac{k}{m} - \frac{\gamma^2}{4}}\ .$$

The $\pm$ up in the exponential means that either the plus or the minus solution is valid. Indeed, since the real part of $e^{i\omega t}$ is identical to the real part of $e^{-i\omega t}$, there's no useful distinction between the two solutions. We might as well just pick one and stick with it.

It is thus simple to make the final solution for the damped harmonic oscillator:

$$\tilde{x}(t) = Ae^{-\gamma t/2} e^{i(\omega t+\phi_0)}$$

so that $x(t) = \mathrm{Re}[\tilde{x}(t)]$ is given by

$$\begin{aligned}x(t) &= Ae^{-\gamma t/2}\cos\left(\omega t + \phi_0\right)\\ \omega &= \sqrt{\frac{k}{m} - \frac{\gamma^2}{4}}\ .\end{aligned}$$

The constants A and ϕ_0 are then set by initial conditions.

I know of no method to solve this equation that is simpler than this complex number technique. It may seem a bit odd to introduce complex numbers to describe totally real phenomena. However, the complex numbers simplify the solution of the differential equations so much that it is *absolutely* worth using them as much as possible.