8.022 (E&M) – Lecture 9

Topics:

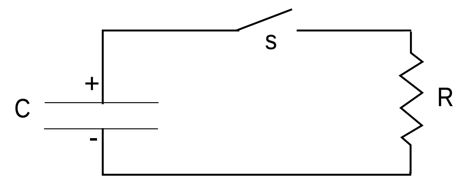
- RC circuits
- Thevenin's theorem

Last time

- Electromotive force:
  - How does a battery work and its internal resistance
- How to solve simple circuits:
  - Kirchhoff's first rule: at any node, sum of the currents in = sum of the currents out (conservation of charge at nodes)
  - Kirchhoff's second rule: around any closed loops, the sum of EMF and potential drops is 0 (electrostatic field is conservative)
- Power dissipated by a resistor: $P = VI = RI^2$

Capacitors in circuits

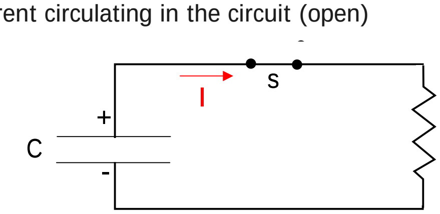
- A new way of looking at problems:
  - Until now: charges at rest or constant currents
  - When capacitors present: currents vary over time



- Consider the following situation:
  - A capacitor C with charge Q_0 → $V_0=Q_0/C$
  - A resistor R in series connected by switch s
- What happens when switch s is closed?

Discharging capacitors: qualitative

- Before switch s is closed:
  - Difference in potential between C plates: V_0
  - No current circulating in the circuit (open)



- After switch s is closed:
  - Difference in potential between capacitor plates will induce current I
  - As I flows, charge difference on capacitor decreases → V_C decreases → I decreases over time

Discharging capacitors: quantitative

- Apply second Kirchhoff's law:
  - EMF supplied by capacitor C: V=Q/C
    - NB: this is true at any moment in time → Q(t) → V(t)
  - Voltage drop on the resistor: -IR

$$\frac{Q}{C} - IR = 0$$

- Not useful in this form since I=I(Q)
  - I=-dQ/dt (- sign because C is losing charge)

$$\frac{Q}{C} + \frac{dQ}{dt}R = 0$$

- Easy integral yields to exponential decay of the charge:

$$Q(t) = Q_0 e^{-\frac{t}{RC}}$$

How to integrate RC circuits

To solve $\frac{Q}{C} + \frac{dQ}{dt}R = 0$, rewrite as: $\frac{dQ}{Q} = -\frac{dt}{RC}$

Integrate both sides:

$$\int_{Q_0}^{Q(t)} \frac{dQ}{Q} = -\int_0^t \frac{dt}{RC}$$

$$\ln \frac{Q(t)}{Q_0} = -\frac{t}{RC}$$

$$\boxed{Q(t) = Q_0 e^{-\frac{t}{RC}}}$$

NB: τ=RC is called "decay constant" of the circuit

Solution of RC circuit

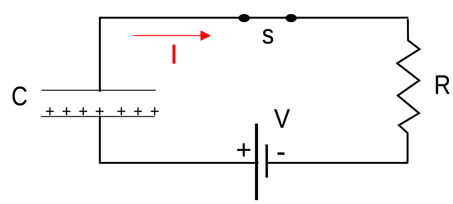
- Solution: $Q(t) = Q_0 e^{-\frac{t}{RC}}$
  - Exponential decay of charge stored in capacitor
  - τ=RC is called "decay constant" of the circuit
  - After a time RC, the charge decreased by 1/e w.r.t. original value
  - Units of RC:
    - cgs: [R]= statvolt s /esu; [C]=esu/statvolt → [RC]=s
    - SI: [R]=V/A; [C]=C/V; A=C/s → [RC]=s
- Derive the current:

$$I(t) = -\frac{dQ}{dt} = -Q_0 \frac{d}{dt}\left(e^{-\frac{t}{RC}}\right) = \frac{Q_0}{RC} e^{-\frac{t}{RC}}$$

  - Same exponential decay as for Q(t)

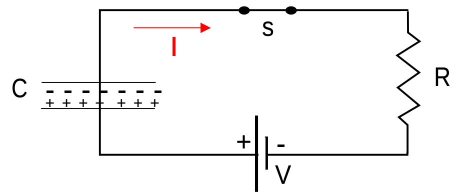
Charging capacitors

- Now 3 elements in circuit: EMF, capacitor and resistor
  - Capacitor starts uncharged



- What happens when switch s is closed?
  - When s is closed, current will suddenly flow and C will charge
  - As C charges, E opposite to EMF builds up and slows down current
  - I(t) stops when V_C reaches V

Charging capacitor: solve the circuit



- Solve using Kirchhoff's second law: $V - \frac{Q}{C} - IR = 0$
  - I(t)=+dQ/dt
  - NB: + because the capacitor is now charging!
- First order differential equation $\frac{dQ}{dt}R + \frac{Q}{C} - V = 0$
- Solution: $Q(t) = CV\left(1 - e^{-\frac{t}{RC}}\right)$

Details of integration

To solve $\frac{dQ}{dt}R + \frac{Q}{C} - V = 0$, rewrite as: $\frac{dQ}{dt} = -\frac{(Q - CV)}{RC}$

Setting: Q'=$Q - CV$

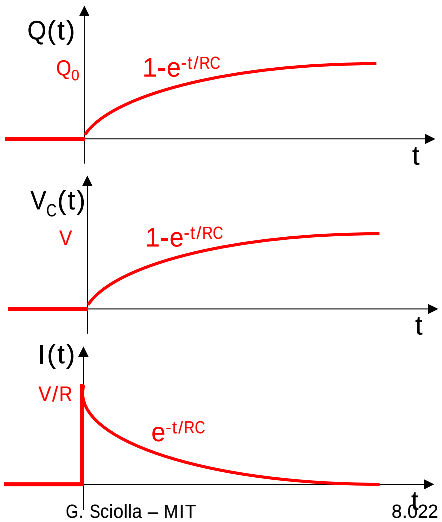
$$\Rightarrow \frac{dQ'}{Q'} = -\frac{dt}{RC}$$

Integrating between t=0 and t:

$$\int_{Q=0}^{Q=Q(t)} \frac{dQ'}{Q'} = -\int_{t=0}^{t=t} \frac{dt}{RC} \Rightarrow \ln \frac{Q(t)\text{-}CV}{\text{-}CV} = -\frac{t}{RC} \Rightarrow \frac{Q(t)\text{-}CV}{CV} = -e^{-\frac{t}{RC}}$$

$$\boxed{Q(t) = CV\left(1 - e^{-\frac{t}{RC}}\right)}$$

Graphical solution



$$Q(t) = CV\left(1 - e^{-\frac{t}{RC}}\right)$$

$$V_C(t) = Q(t)/C = V\left(1 - e^{-\frac{t}{RC}}\right)$$

$$I(t) = \frac{dQ(t)}{dt} = \frac{V}{R}e^{-\frac{t}{RC}}$$

Important comments

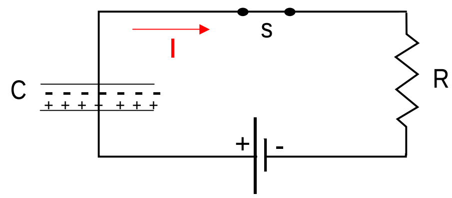
- Solution of RC circuit: $V_C(t) = V\left(1 - e^{-\frac{t}{RC}}\right)$; $I(t) = \frac{V}{R}e^{-\frac{t}{RC}}$
- Are Kirchhoff's laws valid at any moment in time?

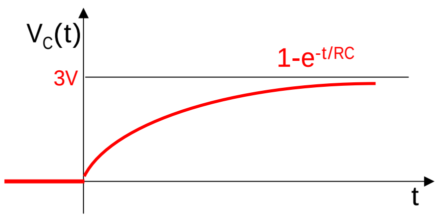
$$V - \frac{Q}{C} - IR = V - V\left(1 - e^{-\frac{t}{RC}}\right) - R\frac{V}{R}e^{-\frac{t}{RC}} = 0 \quad \text{OK!}$$

- Asymptotic behavior of the capacitor:
  - At t=0: I=V/R as if C were a short circuit
  - At t=infinity, I=0 as if C were an open circuit
- Conclusion: no need to solve the differential equation!
  - Solution is an exponential with time constant RC
  - Asymptotic behavior of C gives initial/final values for V(t) and I(t)

Time constant of RC circuit (E9)

- Simple RC circuit with
  - V_{EMF} = 3 V
  - C = 1.3 F
  - R = 11.7 Ω
- Questions:
  - What are V_C and I?
  - Verify that time constant is RC





$$V_C(t) = V_{EMF}\left(1 - e^{-\frac{t}{RC}}\right)$$

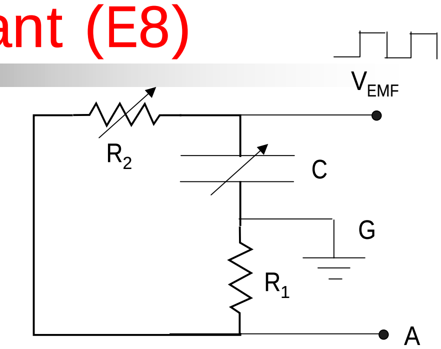
$RC = 15.2s$

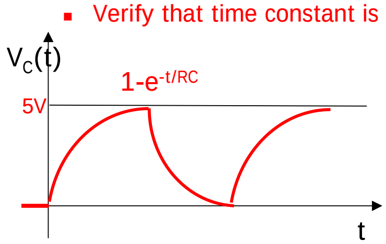
If formula is correct $\Rightarrow$

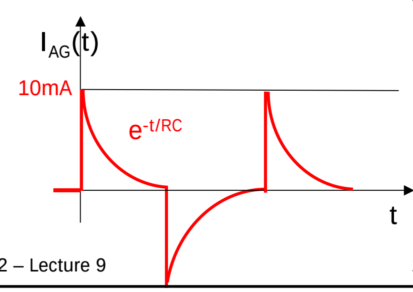
$V_C = V_{EMV}(1-1/e) = 1.9$ V when t=15.2

Verify time constant (E8)

- RC circuit with
  - V_{EMF} = squared 5 V pulses
  - Variable C initially = 0.3 μF
  - Variable R_2 initially = 400 Ω
  - R_1 = 100 Ω
- Display on scope V_C and $I(R_1)$
  - Verify that time constant is RC







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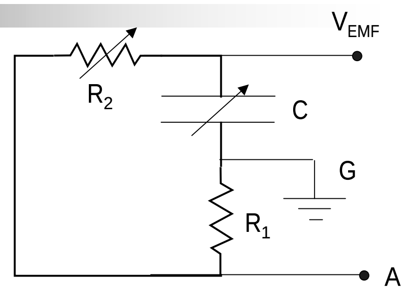
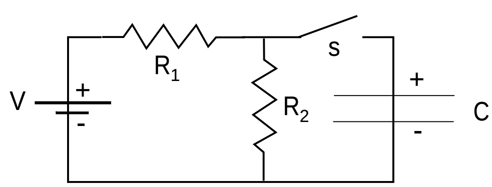


Assuming τ=RC...

- What happens when we double C?
  - τ_1=RC'=2RC=2τ_0 → V (I_{AG}) raises (falls) twice as fast
- How should we change R2 to have the same effect?
  - R'=2R=2(R_1+R_2') → R2': 400 → 900 Ω

More complicated RC circuits

- What if the RC circuit is more than just a series of R and C?
- Consider the following circuit:



- Calculate Q(t) on the capacitor
- Solution:
  - Kirckhoff's laws will solve it: TEDIOUS!
  - Use Thevenin's Theorem

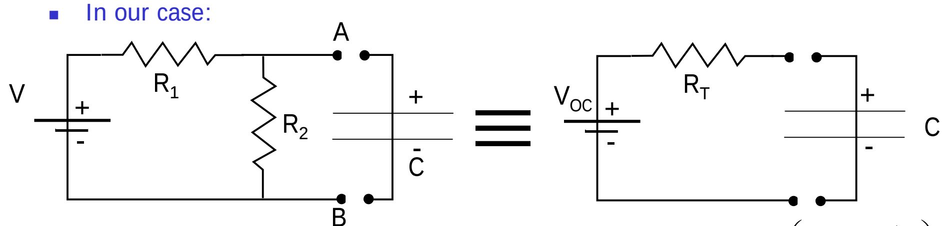
Thevenin equivalence

Thevenin's theorem:

Any combination of resistors and EMFs with 2 terminals can be replaced with a series of a battery V_{OC} and a resistor R_T where

- V_{OC} is the open circuit voltage
- $R_T=V_{OC}/I_{short}$ where I_{short} is the current going through the shorted terminals or $R_T=R_{eq}$ with all the EMF shorted

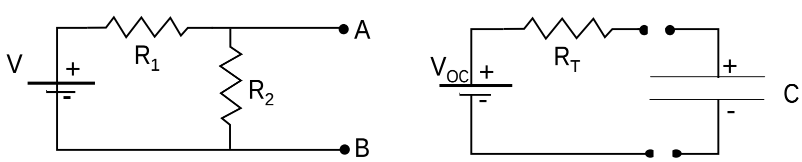
- In our case:



- Once the circuit is reduced, the solution is known: $Q(t)=CV_{OC}\left(1-e^{-\frac{t}{R_T C}}\right)$

Thevenin's demonstration

- Prove that V_{OC} is the open circuit voltage
  - Since $Q(t)=CV_{OC}\left(1-\exp(-\frac{t}{R_T C})\right)$ → $V_C(t)=V_{OC}\left(1-\exp(-\frac{t}{R_T C})\right)$
  - So V_{OC} is the asymptotic V for the capacitor
  - Since for t→ infinity, C→open circuit: V_{OC} = V of the open circuit

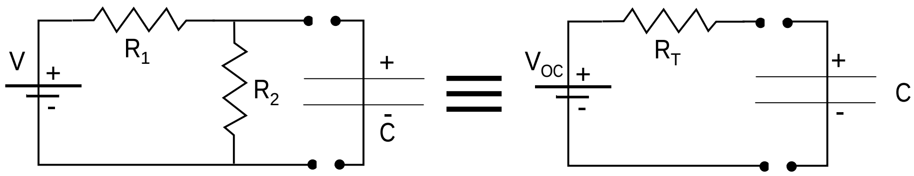


- Prove that $R_T=V_{OC}/I_{short}$ with I_{short}= current through shorted terminals
  - There is only one current going through the reduced circuit
  - At t=0, C behaves like a short → At t=0 $I_{short}=V_{OC}/R_T$

    → $R_T=V_{OC}/I_{short}$

Solve the actual problem

Calculate V_{OC} and $R_T=V_{OC}/I_{short}$ for our problem:



$$\begin{cases} V_{OC} = \dfrac{V}{R_1+R_2}R_2 \\ \text{Shorting C is makes } R_2 \text{ irrelevant in the circuit: } I_{short} = \dfrac{V}{R_1} \\ R_{Thevenin} = \dfrac{V_{OC}}{I_{short}} = \dfrac{R_1R_2}{R_1+R_2} \end{cases}$$

$$\Rightarrow \boxed{Q(t) = C\frac{VR_2}{R_1+R_2}\left(1-e^{-\frac{t(R_1+R)}{CR_1R_2}}\right)}$$

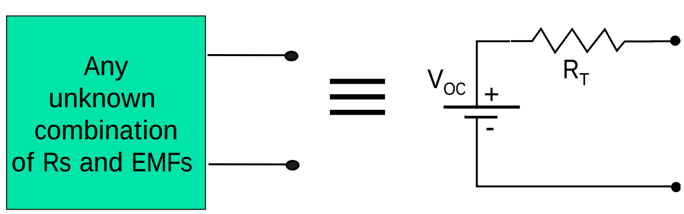
$$\Rightarrow I(t) = \frac{V}{R_1}e^{-\frac{t(R_1+R)}{CR_1R_2}}$$

NB: This is R1//R2, same resistance we would get if we shorted EMF!

Thoughts on Thevenin

The importance of Thevenin:

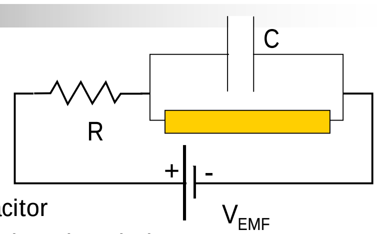
- When we have a messy system or resistors and EMFs, we can reduce it to a simple R+EMF in series just measuring I_{short} and V_{open}:



Careful:

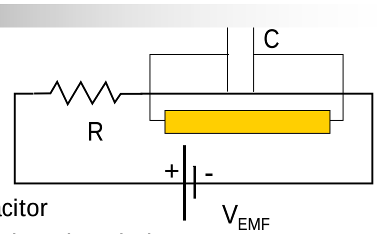
- Thevenin works only when the elements in the box follow Ohm's law, i.e. linear relation between V and I

Oscillating circuit (E13)



- RC circuit with:
  - V_{EMF} = 1 kV
  - C = 0.1 μF
  - R = 2.5 MΩ
  - Fluorescent light in parallel with capacitor
    (R_{FL}<<< R when current flows; ~infinite otherwise)
- Why is light flashing at ν~ 1Hz?
  - Initially the capacitor will start charging (no current through the lamp)
  - When V_C>certain value ~ 1kV → current flows through fluorescent light discharging the capacitor very quickly
  - The process will start again
  - $\nu \sim 1/\tau = 1/RC = 4$ Hz

Oscillating circuit (E13)



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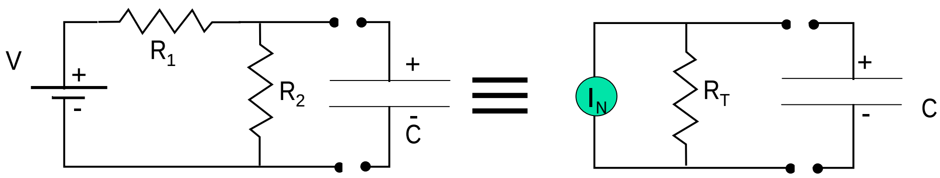
  - V_{EMF} = 1 kV
  - C = 0.1 μF
  - R = 2.5 MΩ
  - Fluorescent light in parallel with capacitor
    (R_{FL}<<< R when current flows; ~infinite otherwise)
- NB: charging and discharging time constants are very different!
  - Charging: fluorescent light is ~ open circuit: $\tau_{charge}=RC$
  - Discharge: fluorescent light has a (very small) resistance R_{FL}
    - Thevenin: $R_T=R//R_{FL}\sim R_{FL}$
    - $\tau_{discharge}=R_TC\sim R_{FL}C<<$

Norton's theorem

Any combination of resistors and EMFs with 2 terminals can be replaced with a parallel of a current generator I_N and a resistor R_T where

- R_T is the equivalent resistance of the circuit with all the EMF shorted and all the current sources open (same as Thevenin!)
- $I_N = V_{OC}/R_T$



$$\begin{cases} R_T = R_1 // R_2 = \dfrac{R_1 R_2}{R_1 + R_2} \\ I_N = \dfrac{V_{OC}}{R_T} = \dfrac{V R_2 / (R_1 + R_2)}{R_1 // R_2} = \dfrac{V}{R_1} \end{cases}$$

Summary and Outlook

- Today:
  - RC circuits
  - Thevenin's theorem
- Next time:
  - Magnetism
- Remember: don't miss office hours
  - Bring your problems and let's find solutions together!