8.022 (E&M) – Lecture 16

Topics:

- Inductors in circuits
  - RL circuits
  - LC circuits
  - RCL circuits

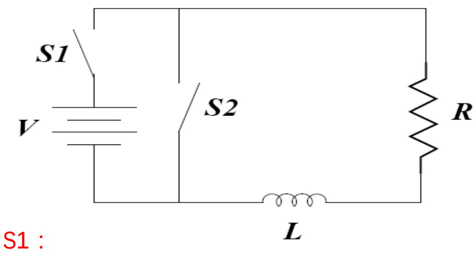
Last time

Our second lecture on electromagnetic inductance

- 3 ways of creating emf using Faraday's law:
  - Change area of circuit S(t)
  - Change angle between B and S → AC generators
  - Change B magnitude
- Self and mutual inductance
  - Energy stored in inductor
  - Applications: transformers

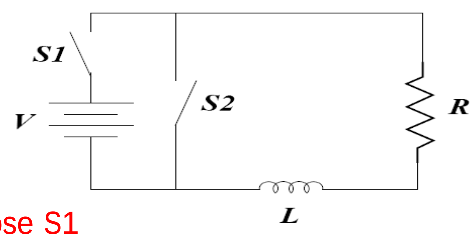
Today is our 3rd lecture on inductance: inductors in circuits

RL circuits: intuitive description



- At t=0, close S1 :
  - Lentz's law opposes change in Φ_B through L
  - Since Φ_B (t=0)= 0, L will impede current flow → I(0)=0
  - As time passes, I will start flowing saturating at I=V/R
- After a long time, simultaneously open S1 and close S2:
  - Lentz's law opposes change in Φ_B through L
  - Back emf will keep current flowing for a while
  - R dissipates power → the current will die exponentially

RL circuits: quantitative description



- At t=0: close S1
  - Kirchoff's rule #2: $V - IR - L\frac{dI}{dt} = 0$

Rewrite as: $-I + \frac{V}{R} = \frac{L}{R}\frac{dI}{dt} \Rightarrow \frac{dI}{I - \frac{V}{R}} = -\frac{R}{L}dt$

$$\Rightarrow \ln\frac{I - V/R}{-V/R} = -\frac{R}{L}t \Rightarrow I - \frac{V}{R} = -\frac{V}{R}e^{-\frac{R}{L}t} \Rightarrow \boxed{I = \frac{V}{R}(1 - e^{-\frac{R}{L}t})}$$

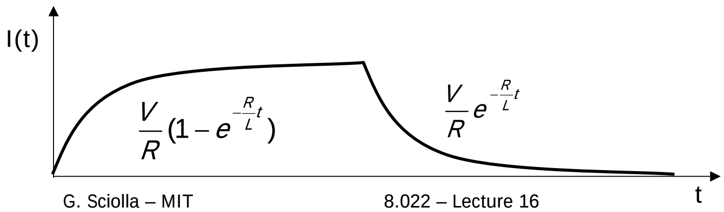
RL circuits: quantitative description(2)

- At t=t': open S1 and close S2
  - Kirchoff's rule #2: $$-IR - L\frac{dI}{dt} = 0$$

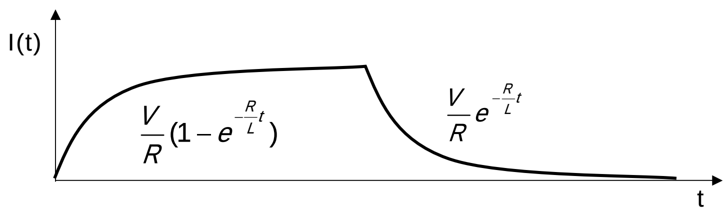
Rewrite as: $$-I = \frac{L}{R}\frac{dI}{dt} \Rightarrow \int_{I=I_0}^{I=I(t)} \frac{dI}{I} = -\int_{t=0}^{t} \frac{R}{L}dt$$

$$\Rightarrow \ln\frac{I}{I_0} = -\frac{R}{L}t \Rightarrow \boxed{I = \frac{V}{R}e^{-\frac{R}{L}t}}$$

- Graphically:



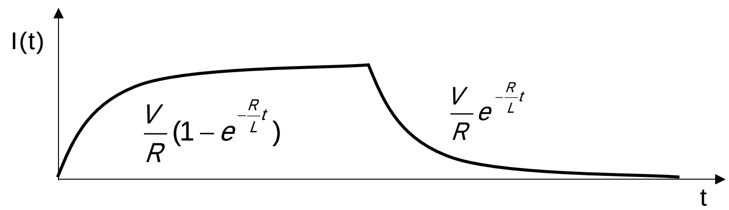
RL circuits: interpretation of results



- How do we interpret these results?
- Inductors cause currents to have an "inertia"
  - If no current flowing: L forces I to build up gradually
  - If current is flowing: L will do what it takes to make it continue (back-emf)
- Asymptotic behavior when "charging" L
  - At t=0, I=0, as if L were an open circuit
  - At t=infinity, I=V/R, as if L did not exist

$$\begin{cases} t=0: & L \to \text{open circuit} \\ t=\infty: & L \to \text{short circuit} \end{cases}$$

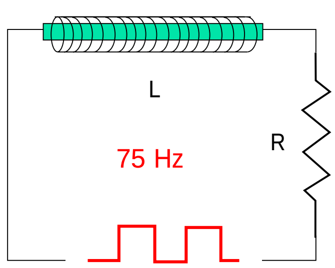
RL circuits: time constant



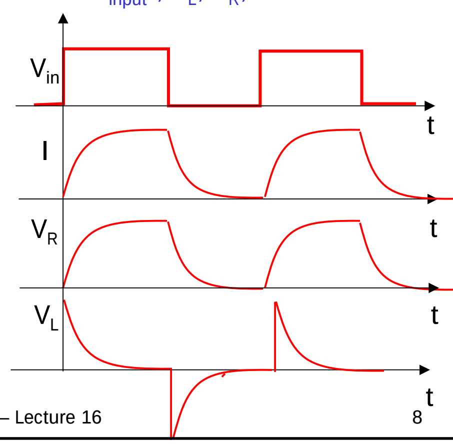
- Results of RL circuit are exponentials, as in RC circuits
  - RC circuit: time constant τ=RC
  - RL circuits: time constant τ=L/R
- NB: time constant is the time it takes the exponential function to decrease (increase) to 1/e (1-1/e) of its original (final) value
- Check units
  - cgs: [L]/[R]=(sec^2/cm)/(sec/cm)=sec
  - SI: [L]/[R]= H/Ω = (V sec/A)/(V/A) = sec

LR time constant

- Consider the following circuit

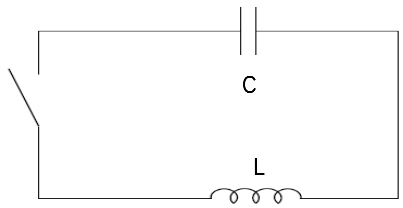


- On the oscilloscope:
  - V_{input} , V_L, V_R, I in the circuit



V_L = L dI/dt

LC circuits



- Start with charged capacitor and close switch at t=0:
  - Kirchoff's second rule: $\frac{Q}{C} - L\frac{dI}{dt} = 0$

    Since $I = -\frac{dQ}{dt}$: $\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$
  - How to solve this? Educated guess: $Q(t) = A\cos\omega_0 t + B\sin\omega_0 t$

    $\Rightarrow \frac{d^2Q}{dt^2} = -\omega_0^2 A\cos\omega_0 t - \omega_0^2 B\sin\omega_0 t = -\omega_0^2 Q(t) \Rightarrow$ $\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$

LC circuits: solution

- Plug this in the differential equation:

$$\frac{d^2Q(t)}{dt^2} = -\frac{1}{LC}Q(t) \Rightarrow -\omega_0^2 Q(t) = -\frac{1}{LC}Q(t) \Rightarrow \boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

- Determine constants A and B from initial conditions:
  - $Q(t=0) = Q_0 = A\cos(0) + B\sin(0)$ → $A = Q_0$
  - $I(t=0) = 0 = -\omega_0 A\sin(0) + \omega_0 B\cos(0)$ → $B = 0$
- Complete solution:

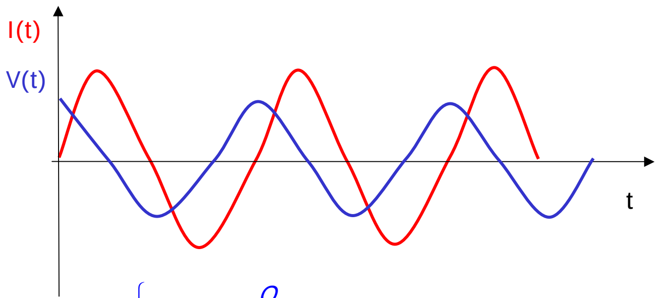
$$Q(t) = Q_0 \cos\omega_0 t \Rightarrow V_C(t) = \frac{Q(t)}{C} = \frac{Q_0}{C}\cos\omega_0 t$$

$$I(t) = -\frac{dQ}{dt} = \frac{Q_0}{\sqrt{LC}}\sin\omega_0 t$$

  - NB: current and voltages are off by 90 degrees

LC circuits: solution

Graphical representation of the solution:



$$\begin{cases} V_c(t) = \dfrac{Q_0}{C}\cos\omega_0 t \\ I(t) = \dfrac{Q_0}{\sqrt{LC}}\sin\omega_0 t \end{cases}$$

NB: Q and I have a phase of 90 deg

Energy conservation

- Energy stored in the capacitor over time:

$$U_C(t) = \frac{Q^2(t)}{2C} = \frac{Q_0^{\,2}(t)}{2C}\cos^2\omega_0 t$$

- Energy stored in the inductor:

$$U_L(t) = \frac{1}{2}LI(t)^2 = \frac{1}{2}L\frac{Q^2_{\ 0}}{LC}\ \sin^2\omega_0 t = \frac{Q_0^{\,2}}{2C}\sin^2\omega_0 t$$

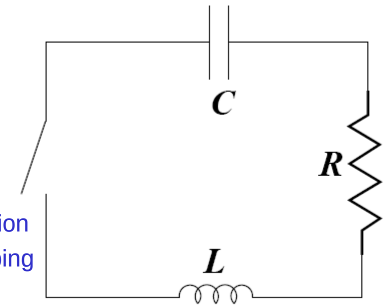
- Total energy:

$$U(t) = U_L(t) + U_C(t) = \frac{Q_0^{\,2}}{2C}(\sin^2\omega_0 t + \cos^2\omega_0 t) = \frac{Q_0^{\,2}}{2C}$$

- What is happening over time?
  - Energy swings back and forth between C and L but at any moment in time the total energy is equal to the energy initially stored in the capacitor: Energy is conserved!

RCL circuits

- LC circuits don't belong to this world:
  - R is never exactly 0!
- So let's concentrate on RCLs
  - Start with a charged C
- Intuitively:
  - LC → oscillatory part: sin and cos solution
  - R → dissipative part: exponential damping
- Rigorous solution:



Use Kirchoff: $\frac{Q}{C} - IR - L\frac{dI}{dt} = 0$

Since $I(t) = -\frac{dQ}{dt} \Rightarrow \frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0$

RCL circuits: solution

- How to solve this equation? $\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0$
- Educated guess!
  - Intuition tells us that the solution must have an oscillatory term and a damping term
    - Strategy #1: exponential * sin/cos functions:

      $$Q(t) = e^{-t/\tau}\left(A\cos\omega_0 t + B\sin\omega_0 t\right)$$

      Very heavy on algebra!!!
    - Strategy #2: complex exponentials
      - Idea: the solution is the real part of a complex solution

      $$\tilde{Q}(t) = A\, e^{i\phi_0} e^{i\alpha t} \Rightarrow Q(t) = \text{Re}\left[\tilde{Q}(t)\right]$$

      Much easier algebra!!!

      NB: a can be complex!

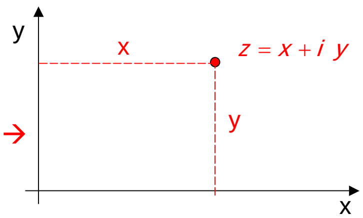
See handout on complex number + sections next week

Complex number notation

- Complex number: number with both a real and an imaginary part

  $z = x + i\,y$ with $i=\sqrt{-1}$

- Complex plane representation z=(x,y) →



- Another useful representation

  Set magnitude $r=\sqrt{x^2+y^2}$ and phase $\theta=\text{arctg}\dfrac{y}{x} \Rightarrow z = r(\cos\theta + i\sin\theta)$

- Given Euler's relation: $e^{i\theta} = \cos\theta + i\sin\theta$
  - Prove it using Maclaurin expansion (see handout)

  $\Rightarrow\ z = re^{i\theta}$ (Phasor representation)

RCL circuits: solution (cont)

Plug expected solution $\tilde{Q}(t) = e^{i\phi_0} e^{i\alpha t}$ into the differential equation

$$\frac{d^2\tilde{Q}}{dt^2} + \frac{R}{L}\frac{d\tilde{Q}}{dt} + \frac{1}{LC}\tilde{Q} = 0$$

$$\frac{d\tilde{Q}}{dt} = i\alpha Q;\quad \frac{d^2\tilde{Q}}{dt^2} = -\alpha^2\tilde{Q} \quad\Rightarrow\quad \tilde{Q}\left(-\alpha^2 + i\alpha\frac{R}{L} + \frac{1}{LC}\right) = 0$$

Simple quadratic equation: $-\alpha^2 + i\alpha\dfrac{R}{L} + \dfrac{1}{LC} = 0 \quad\Rightarrow\quad \alpha = i\dfrac{R}{2L} \pm \sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}$

This gives us 2 complex solutions for $\tilde{Q}(t)$:

$$\begin{cases} \tilde{Q}_+(t) = Ae^{i\phi_0} e^{-\frac{R}{2L}t} e^{i\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\,t} \\ \tilde{Q}_-(t) = Ae^{i\phi_0} e^{-\frac{R}{2L}t} e^{-i\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\,t} \end{cases}$$

$\Rightarrow$ real part: $Q(t) = Ae^{-\frac{R}{2L}t}\cos(\pm\omega t + \phi_0)$ with $\omega=\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}$

The weak damping limit

Weak damping limit: small R→ the damping is small → several oscillations occur before amplitude start decreasing in sizable way

$$I(t) = -\frac{dQ}{dt} = Q_0 e^{-\frac{R}{2L}t}[\omega \sin(\omega t + \phi_0) + \frac{R}{2L}\cos(\omega t + \phi_0)\ ]$$

When $\omega >> R/(2L)$ (damping limit), the second term can be ignored and

$$I(t) \sim A e^{-\frac{R}{2L}t} \omega \sin(\omega t) \text{ with } \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \sim \frac{1}{\sqrt{LC}} = \omega_0$$

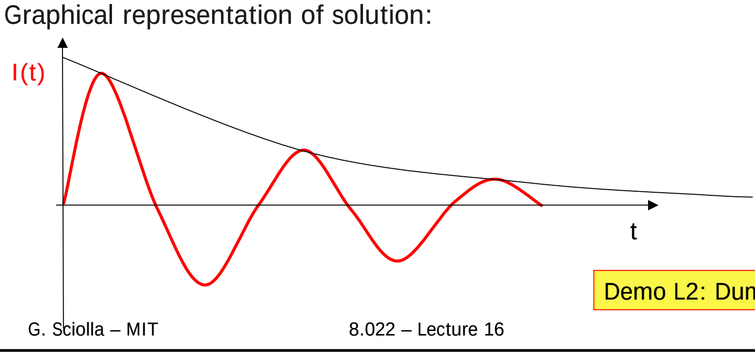
$$\Rightarrow \text{final solution for "weak damping": } \begin{cases} Q(t) \sim Q_0 e^{-\frac{R}{2L}t} \cos(\omega_0 t + \phi_0) \\ I(t) \sim \omega_0 Q_0 e^{-\frac{R}{2L}t} \sin(\omega_0 t + \phi_0) \\ \omega_0 = \frac{1}{\sqrt{LC}} \end{cases}$$

RCL in weak damping limit

- Initial conditions: $Q(0)=Q_0=A\cos(\phi_0)$ and $I(0)=0=A\omega_0\sin\phi_0 \Rightarrow A = Q_0;\ \phi_0 = 0$

$$\Rightarrow \begin{cases} Q(t) \sim Q_0 e^{-\frac{R}{2L}t} \cos(\omega_0 t) \\ I(t) \sim \omega_0 Q_0 e^{-\frac{R}{2L}t} \sin(\omega_0 t) \end{cases}$$

- Graphical representation of solution:



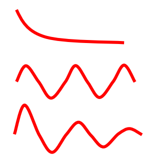
Demo L2: Dumped RCL

Summary and outlook

- Today:

  What happens when we put L in circuits?

  - RL circuits: exponential solutions
  - LC circuits: oscillatory solution
  - RCL circuits: damped oscillation



- Next Tuesday:
  - Quiz # 2: good luck!!!