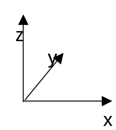
8.022 (E&M) – Lecture 12

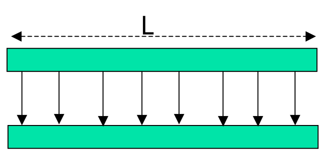
Topics:

- Continuation of Special Relativity
  - Transformation of Electric Fields
  - Relativistic Momentum and Energy
  - Transformation of Forces
  - Prove that E and B are equivalent in different reference frames

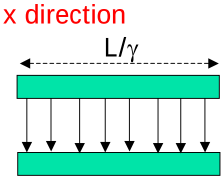
Electric Fields in Motion



- Consider a parallel plate capacitor
  - Squared plates of side L in the xy plane
  - Charge Q distributed on the plates:
    - $\sigma = Q/L^2$
  - Electric field // z axis:
    - $E = 4\pi\sigma = 4\pi Q/L^2$



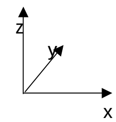
- The capacitor is now boosted with velocity **v** in the x direction
  - How does E transform?
  - $E' = 4\pi\sigma' = 4\pi Q'/A' = 4\pi Q/LL' = 4\pi Q/L(L/\gamma) = \gamma\, 4\pi Q/A = \gamma E$



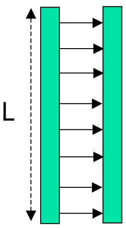
- Conclusion: $E'_{\perp} = \gamma E$

- Expected result: the field lines get thicker as L shrinks…

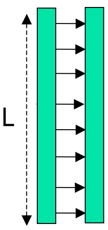
Electric Fields in Motion (2)



- Now orient the capacitor with the plates in the yz plane:
  - Charge Q distributed on the plates:
    - $\sigma = Q/L^2$
  - Electric field // z axis:
    - $E = 4\pi\sigma = 4\pi Q/L^2$



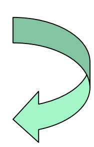
- Boost again the capacitor with velocity **v** in the x direction
  - How does E transform?
  - $E' = 4\pi\sigma' = 4\pi Q'/A' = 4\pi Q/L'_y L'_z = 4\pi Q/A = E$



- Conclusion: $$E'_{//} = E$$
- Expected result: the field lines keep the same distance...
- How does the capacitance change?

Momentum and Energy

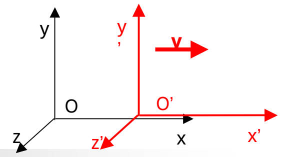
- For a particle of mass m moving with velocity **u**
- Classical definitions:
  - Momentum: $\vec{p} = m\vec{u}$
  - Kinetic energy: $E_{kin} = \frac{1}{2} m u^2$
- Relativistic definition
  - Momentum: $\vec{p} = \gamma_u m\vec{u}$
  - Energy: $E = \gamma_u m c^2$
    - where γ_u is the relativistic γ factor: $\gamma_u = 1/(1-u^2/c^2)$



Why? See handout #2

- For low velocities, the new formulae reproduce old ones (Taylor!)
  - $\vec{p} = \gamma_u m\vec{u} \sim (1 + \frac{1}{2}\frac{u^2}{c^2} - ...) m\vec{u} \sim m\vec{u}$
  - $E_{kin} = \gamma_u m c^2 \sim (1 + \frac{1}{2}\frac{u^2}{c^2} - ...) m c^2 \sim m c^2 + \frac{1}{2} m u^2$

Transformation of p and E



- Consider 2 inertial reference frames: O and O'
  - O' is moving w.r.t. O with velocity **v** // x axis
- How do momentum and energy Lorentz transform?
  - One can demonstrate that

$$\begin{cases} E' = \gamma_v (E - \beta_v c p_x) \\ p'_x = \gamma_v (p_x - \beta_v E / c) \\ p'_y = p_y \\ p'_z = p_z \end{cases}$$

    - Sorry not today: no time!
  - Where $\gamma_u = 1/(1-\beta_v^2)$ and $\beta_v^2 = v^2/c^2$

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Transformation of Forces

- In an inertial R.F. O a force F_x is acting on a body of mass m
  - Body is initially at rest: p=0 at t=0
  - Small acceleration → non relativistic velocities involved in O
    - Force F // to x axis: $F_x = \frac{dp_x}{dt}$
    - Change in position: $\Delta x = \frac{1}{2} a \Delta t^2 = \frac{1}{2} \frac{F_x}{m} \Delta t^2$
    - Change in Energy: $\Delta E = \frac{(\Delta p)^2}{2m} = \frac{(F_x \Delta t)^2}{2m}$
- How do these quantities look like in the Lab Frame O'?
  - NB: O' is moving with velocity **v**//x axis wrt the Frame O

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Forces // v

- How does F_x look in the Lab Frame O' ?

$$F'_x = \frac{dp'_x}{dt'}$$

Remember Lorentz transformations: $E' = \gamma_v (E - \beta_v c p_x)$ and $p'_x = \gamma_v (p_x - \beta_v E / c)$

$$F'_x = \frac{\Delta p'_x}{\Delta t'} = \frac{\Delta\left[\gamma(p_x - \beta E / c)\right]}{\Delta\left[\gamma\left(t - \frac{v}{c^2}x\right)\right]} = \frac{\Delta p_x - \beta \Delta E / c}{\Delta t - \frac{v}{c^2}\Delta x} = \frac{F_x - \beta / c \frac{F_x^2 \Delta t}{2m}}{1 - \frac{v}{c^2}\frac{1}{2}\frac{F_x}{m}\Delta t}$$

For $\Delta t \to 0$, this becomes: $F'_x = \lim_{\Delta t \to 0} F'_x = \lim_{\Delta t \to 0} \frac{F_x - \beta / c \frac{F_x^2 \Delta t}{2m}}{1 - \frac{v}{c^2}\frac{1}{2}\frac{F_x}{m}\Delta t} = F_x$

- Conclusion:
  - The component of the force // to $\mathbf{v}_{RF}$ is constant: $F'_x = F_x$

Forces perpendicular to v

- How does F_y look in the Lab Frame O' ?

$$F'_y = \frac{dp'_y}{dt'}$$

Remember Lorentz transformations: $E' = \gamma_v (E - \beta_v c p_x)$ and $p'_y = p_y$

$$F'_y = \frac{\Delta p'_y}{\Delta t'} = \frac{\Delta p_y}{\Delta\left[\gamma\left(t - \frac{v}{c^2}x\right)\right]} = \frac{\Delta p_y}{\gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)} = \frac{F_y}{\gamma\left(1 - \frac{v}{c^2}\frac{1}{2}\frac{F_x}{m}\Delta t\right)}$$

For $\Delta t \to 0$, this becomes: $F'_y = \lim_{\Delta t \to 0} F'_x = \lim_{\Delta t \to 0} \frac{F_y}{\gamma\left(1 - \frac{v}{c^2}\frac{1}{2}\frac{F_x}{m}\Delta t\right)} = \frac{F_y}{\gamma}$

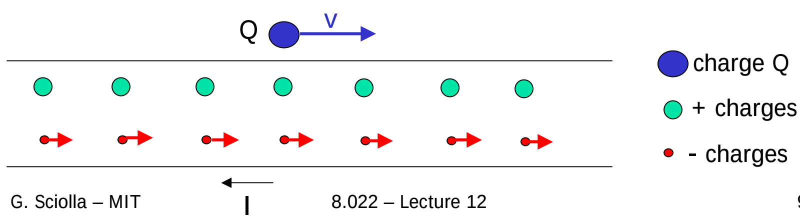
- Conclusion:
  - Components of force perpendicular to $\mathbf{v}_{RF}$ are contracted: $F'_y = \frac{F_y}{\gamma}$

Is this consistent with what we found about E?

Please pay attention: this is difficult!

Force by current on moving charge

- Description in the Lab Frame
  - Electrically neutral wire carrying a current
  - Positive charge density $\lambda_+ = \lambda_+^{REST} = \lambda_0$.
    - NB: these charges are at rest in O'
  - Negative charge density $\lambda_- = \lambda_-^{MOT} = -\lambda_0$
    - NB: these charges are moving with velocity **u**
    - → $-\lambda_0$ is not the density of the electrons in their reference frame O
  - A charge Q outside the wire moves to right with velocity **v**



λ of negative charges in their RF

- We said: $\lambda_- = \lambda_-^{MOT} = -\lambda_0$
- What is charge density in their RF? λ_0^{REST}
- First attempt:
  - Charge density λ_0 = Q/L where L = length of the wire in Lab frame
  - In lab frame: λ_-^{MOT} = Q/L= $-\lambda_0$
  - In O (in rest with - charges), length of wire appears contracted: L':L/γ

  → $\lambda_-^{REST} = Q/L' = Q\gamma/L = -\gamma\lambda_0$ → $\lambda_-^{REST} > \lambda_-^{MOT}$ WRONG!

  Why? There is no such thing as the wire. Just the length of + and – charges which happen to be the same in the Lab reference frame but not elsewhere.
- Second attempt:
  - The electrons will think: our length in our own RF is L'₋. In the reference frame of the lab, boosted wrt us by a velocity –v, this length will be contracted by a factor γ: $L'_- = \gamma L$

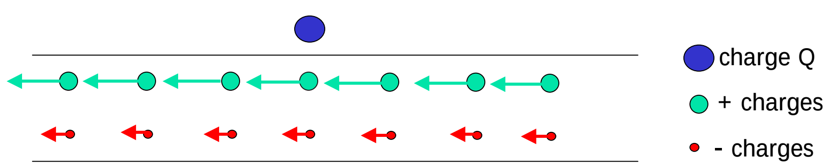
  → $\lambda_-^{REST} = Q/L'_- = Q/\gamma L = -\lambda_0/\gamma$ → $\lambda_-^{REST} < \lambda_-^{MOT}$

Force by current on moving charge

- What forces act on the charge Q? Lab frame:
  - Wire is neutral: no electric filed E
  - Current will generate magnetic filed B:
    - Current in the wire: I=dQ/dt=$\lambda_0$dx/dt=λ_0u
    - Ampere's law: $B = \frac{2I}{cr} = \frac{2\lambda_0 u}{cr}$
  - Magnetic force acting on charge Q: $F = Q\frac{v}{c}B = Q\frac{2\lambda_0 uv}{c^2 r}$
  - Direction?
    - Right hand rule: repulsive force
    - NB: I opposite to v electrons!

Force in charge's rest frame?

- Let's now move to the charges rest frame:



- Velocities involved:
  - Charge Q: at rest by definition
  - Negative charges in the wire: velocity u'=(u-v)$_{\text{relativistic sum}}$ $u' = \frac{u-v}{1-uv/c^2}$
  - Positive charges in the wire: velocity –v
- Is there any force acting on Q?
  - There must be: Relativity Principle!
  - No magnetic force: the charge is at rest!

Charge densities in Q's RF

- Are we in trouble?
- Let's see what happens to the charges in the wire
- Positive charges:
  - Charge density in charge's reference frame: $\lambda_+'=Q/L'=\gamma_v\lambda_0$
- Negative charges:

$$\lambda_-'=\gamma_{u'}\lambda_-^{REST}=\gamma_{u'}\frac{-\lambda_0}{\gamma_u}=\gamma_u\gamma_v(1-\beta_u\beta_v)\frac{(-)}{\gamma_u}=-\gamma_v(1-\beta_u\beta_v)\lambda_0$$

Goal: calculate $\gamma_{u'}$. Let's start calculating $1/\gamma^2_{u'}=1-\beta^2_{u'}$

$$1-\beta^2_{u'}=1-\left(\frac{u'}{c}\right)^2=1-\frac{\left(\frac{u-v}{1-uv/c^2}\right)^2}{c^2}=1-\left(\frac{\beta_u-\beta_v}{1-\beta_u\beta_v}\right)^2=\frac{1-2\beta_u\beta_v+\beta_u^2\beta_v^2-(\beta_u-\beta_v)^2}{(1-\beta_u\beta_v)^2}=$$

$$=\frac{(1-\beta_v^2)(1-\beta_u^2)}{(1-\beta_u\beta_v)^2}=\frac{1}{\gamma^2_u\gamma^2_v(1-\beta_u\beta_v)^2}\quad\Rightarrow\quad\boxed{\gamma_{u'}=\gamma_u\gamma_v(1-\beta_u\beta_v)}$$

Force in charge's rest frame

- Net charge density in Q's reference frame:

$$\lambda'_{NET}=\lambda_+'+\lambda_-'=\gamma_v\lambda_0-\gamma_v(1-\beta_u\beta_v)\lambda_0=\gamma_v\beta_u\beta_v\lambda_0=\gamma_v\frac{uv}{c^2}\lambda_0$$

- In this RF there is a net charge on the wire! → Electric field!

$$E'=\frac{2\lambda'_{NET}}{r}=\gamma_v\frac{2uv\lambda_0}{rc^2}$$

- Electric field → force F' will act on the charge Q

$$F'=QE'=\gamma_v\frac{2Quv\lambda_0}{rc^2}\quad\text{(repulsive)}$$

- Is there a Magnetic field as well?
  - Yes, but it does not excerpts any force on Q because Q is at rest

Comparison of forces in the 2 RFs

- In lab frame:
  - Repulsive magnetic force acting on charge Q:

$$F = Q\frac{v}{c}B = Q\frac{2\lambda_0 uv}{c^2 r}$$

- In Q's rest frame:
  - Repulsive electric force acting on charge Q:

$$F' = QE' = \gamma_v Q\frac{2\lambda_0 uv}{c^2 r}$$

- Are results consistent?
  - Yes! We have seen that forces in direction perpendicular to **v** transform as

$$F'_y = \frac{F_y}{\gamma}$$

Thoughts on this problem

- Is the comparison fair?
  - In one RF we have a magnetic force, in the other an electric force
  - Are we comparing apples and oranges?
- No, on the contrary!
  - This results proves that Electricity and Magnetism are intimately connected!
- Physics is consistent!
  - Principle of relativity demands that the 2 observers will come to the same conclusions
  - The details of the calculation (Electric? Magnetic?) are different in the different RF, but ultimately irrelevant.

Summary of Special Relativity

- Speed of light and physics are the same in all RF
- Consequences in mechanics
  - Time dilation
    - Moving clocks run slower $\Delta t' = \gamma \Delta t$
  - Length contraction
    - Moving objects appear shorter along direction of motion: $\Delta L = \gamma \Delta L'$
  - Force transformation
    - Components // v: constant; perpendicular to v: contracted: $F'_y = \frac{F_y}{\gamma}$
- Consequences in E&M
  - Pure B in one RF looks like E in another
  - And vice versa, pure E in one RF looks like E+B in another
    - Because there is a force even in the particle's reference frame

Outlook

- Today:
  - Conclusion of Introduction to Special Relativity
    - Transformations for momentum, energy and forces
    - Proved that E and B are intimately connected
      - Two observers, "relativistically" consistent results
- Next time:
  - Back to Magnetism
    - Ampere's law, Biot-Savart, Vector potential