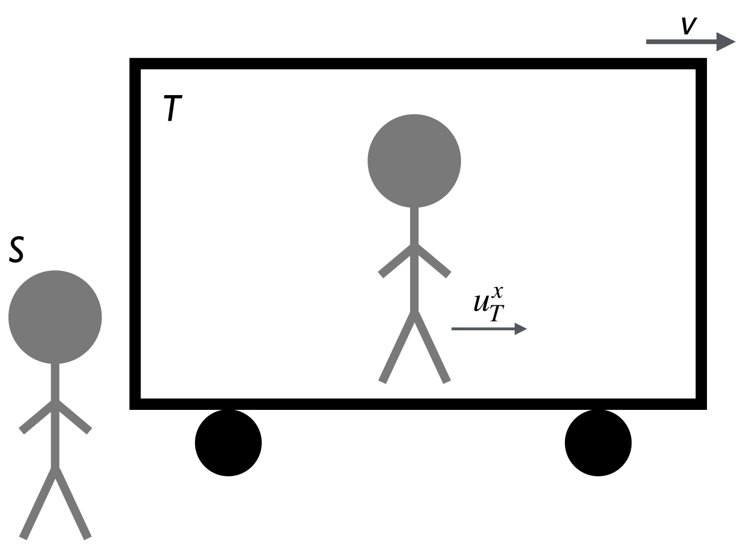
Massachusetts Institute of Technology
Department of Physics
8.033 Fall 2024

Lecture 6
Kinematics in spacetime

6.1 Transforming velocities

With what we've done so far, we've started to develop a good understanding of length, time, and geometry in spacetime. This is a good start for us to begin understanding physics in special relativity, but it's just a start.

In this lecture, we start examining *kinematics* — the properties of moving bodies, and how these properties transform between different reference frames. We begin by looking at velocity. Consider frame T, tied to a train, and consider a person walking inside that train. This train is moving with velocity $\mathbf{v} = v\mathbf{e}_x$ as seen by an observer who is at rest in the station frame S. The person who is walking inside the train is seen to walk with speed u^x_T, also in the x direction, by an observer who is at rest in frame T. (Comment: we will try as much as possible to use the letter u to stand for speeds inside a particular frame; we will try to use v to describe the speeds and velocities that relate two different frames.)



What is the speed u^x_S that observers in frame S measure? In Newtonian physics, we would just add the velocities in frame T to the velocity that frame T has relative to S. To get u^x_S in a world in which all observers agree that light moves at speed c, we work this out using the Lorentz transformation. On the train, we know that in a time interval Δt_T observer T moves through a distance $\Delta x_T = u^x_T \Delta t_T$. Both the time interval and the space interval are affected by the transformation:

$$\begin{aligned} u^x_S = \frac{\Delta x_S}{\Delta t_S} &= \frac{\gamma(\Delta x_T + v\Delta t_T)}{\gamma(\Delta t_T + v\Delta x_T/c^2)} \\ &= \frac{(\Delta x_T/\Delta t_T + v)}{(1 + \frac{v\Delta x_T}{c^2\Delta t_T})} \\ &= \frac{u^x_T + v}{1 + u^x_T v/c^2} \; . \end{aligned} \tag{6.1}$$

This formula has an interesting consequence: using it, we can prove that we can never add sub-light speeds to get a speed that exceeds the speed of light. You will work this out in detail on a problem set, but to see the general idea, imagine that $u^x_T = v = 0.9c$. Then,

$$u^x_S = \frac{0.9c + 0.9c}{1 + (0.9c)(0.9c)/c^2} = \frac{1.8c}{1.81} = 0.9945c \; . \tag{6.2}$$

How do components of the velocity perpendicular to the frames' relative motion transform? Imagine that the person on the train has motion along the y direction as well, so that in Δt_T they move through $\Delta y_T = u^y_T \Delta t_T$. Then,

$$\begin{aligned} u^y_S = \frac{\Delta y_S}{\Delta t_S} &= \frac{\Delta y_T}{\gamma(\Delta t_T + v\Delta x_T/c^2)} \\ &= \frac{u^y_T}{\gamma(1 + u^x_T v/c^2)} \; . \end{aligned} \tag{6.3}$$

(Note that the factor $\gamma = 1/\sqrt{1 - v^2/c^2}$ — it only depends on the relative speed v of the two frames, it does not involve the velocity $\mathbf{u}$.) If the person on the train has velocity along the z direction, then it transforms like Eq. (6.3) as well, replacing u^y with u^z.

6.2 Momentum I: Did we break physics???

A lesson of the previous section is that how velocities add is "weird" as compared to Newtonian expectations. These expectations follow the logic of Galilean relativity, so it should not be too surprising that things change when we impose the rule that c is the same to all observers. However, our laws of classical mechanics have implicitly assumed Galilean relativity. What happens to important principles like conservation of momentum when we "update" our rules for how velocities add?

Let us first review how conservation of momentum works in Newtonian physics. Suppose that we have N_i bodies that come together in some fashion, interact, and then have N_f bodies in the final state. Conservation of momentum tells us that

$$\sum_{j=1}^{N_i} m_j^{\text{initial}} \mathbf{u}_j^{\text{initial}} = \sum_{j=1}^{N_f} m_j^{\text{final}} \mathbf{u}_j^{\text{final}} \; . \tag{6.4}$$

As long as

$$\sum_{j=1}^{N_i} m_j^{\text{initial}} = \sum_{j=1}^{N_f} m_j^{\text{final}} \; , \tag{6.5}$$

this relation holds in all Galilean reference frames.

Let's take a look at what happens when we examine this law in Lorentzian reference frames. Let's consider something really simple: two particles, A and B, of identical mass that collide and rebound elastically. We assume that it remains the case that $\mathbf{p} = m\mathbf{u}$, and begin by examining this situation in the *center of momentum* frame, i.e. the frame in which the net momentum of the system is zero:

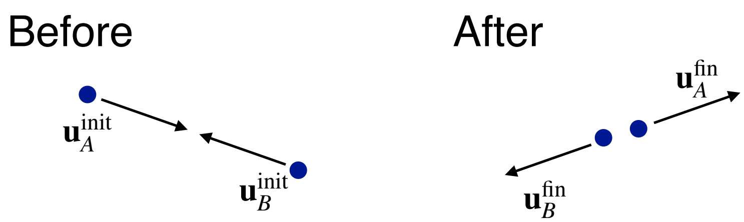


Figure 1: Elastic collision of identical bodies in the center of momentum frame.

The bodies' velocities are given by $\mathbf{u}_A^{\rm init} = u^x \mathbf{e}_x - u^y \mathbf{e}_y$, $\mathbf{u}_B^{\rm init} = -u^x \mathbf{e}_x + u^y \mathbf{e}_y$ before the collision. Afterwards, we have $\mathbf{u}_A^{\rm fin} = u^x \mathbf{e}_x + u^y \mathbf{e}_y$, $\mathbf{u}_B^{\rm init} = -u^x \mathbf{e}_x - u^y \mathbf{e}_y$. Because $m_A = m_B$, we can see that momentum is clearly conserved: It is zero both before and after the collision.

Let's next examine this from another frame of reference. Suppose we examine this collision from a frame that moves with velocity $\mathbf{v} = -u^x \mathbf{e}_x$ with respect to the center of momentum frame. The horizontal motion of particle B is canceled out here; if we are in this frame, we are essentially jogging along with particle B.

What are the velocity vectors in this frame? To find out, we use the relativistic velocity addition formulas we just worked out to get the components of these vectors. Let's do the x components first:

$$u_A^{x'} = \frac{u^x + u^x}{1 + (u^x)^2/c^2} = \frac{2u^x}{1 + (u^x)^2/c^2} \;, \tag{6.6}$$

$$u_B^{x'} = \frac{u^x - u^x}{1 - (u^x)^2/c^2} = 0 \;. \tag{6.7}$$

Notice in this frame, the horizontal velocity components are not equal and opposite, and so the system must have some non-zero horizontal momentum component. This is not surprising: we've moved into a frame in which the entire system is moving in the $+x$ direction, so we expect the system to have momentum along x.

Next, look at the y components:

$$u_A^{y'} = -\frac{u^y}{\gamma(1 + (u^x)^2/c^2)} = -\frac{u^y\sqrt{1 - (u^x)^2/c^2}}{1 + (u^x)^2/c^2} \;, \tag{6.8}$$

$$u_B^{y'} = \frac{u^y}{\gamma(1 - (u^x)^2/c^2)} = \frac{u^y}{\sqrt{1 - (u^x)^2/c^2}} \;. \tag{6.9}$$

The γ that we use here is the one corresponding to the velocity of this frame relative to the center of momentum frame: $\mathbf{v} = -u^x \mathbf{e}_x$, and so $\gamma = 1/\sqrt{1 - (u^x)^2/c^2}$.

Notice that the velocity components in the vertical direction are **no longer equal and opposite**. This means that they do not balance out, and so the system has net momentum in the vertical direction. In other words, under the hypothesis that momentum $\mathbf{p} = m\mathbf{u}$, we appear to have a problem: *The system appears to have acquired momentum in the y direction by moving into a new frame that is moving in the $-x$ direction with respect to the center of momentum frame.*

Our hypothesis that c is the same to all observers, which led to our new velocity addition rules, appears to have broken momentum.

6.3 Momentum II: From Newtonian momentum to Einsteinian momentum

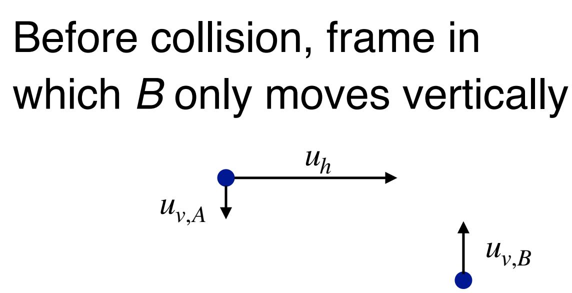
This appears disturbing. However, we have already seen (and you examined on a pset) that the Lorentz transformations are *approximately* consistent with Galilean coordinate transformations. Galilean relativity (and thus Newtonian physics) works fine when speeds are far smaller than c. Perhaps the root cause of this disturbing apparent breakdown is that Newtonian momentum (which respects Galilean relativity) is itself an approximation to a more "Lorentzian" quantity.

Let us try the hypothesis that momentum is defined by

$$\mathbf{p} = \alpha(u) m \mathbf{u} \ . \tag{6.10}$$

The function $\alpha(u)$ is a function corrects the magnitude of momentum, and only depends on the magnitude of the body's velocity $\mathbf{u}$.

With this in mind, we re-examine the collision from the Lorentz frame in which particle B has no horizontal motion:



To simplify some of the analysis which will follow later, we've introduced new labels for the velocity components of these bodies: u_h is the horizontal velocity component of body A in this frame; $u_{v,A}$ is the vertical velocity component of A in this frame; and $u_{v,B}$ is the vertical velocity component of B in this frame. Comparing to our previous calculations given in Eqs. (6.6), (6.8), and (6.9), these velocity components according to the relativistic velocity addition formula are given by

$$u_h = \frac{2u^x}{1 + (u^x)^2/c^2} \ , \quad u_{v,A} = -\frac{u^y\sqrt{1 - (u^x)^2/c^2}}{1 + (u^x)^2/c^2} \ , \quad u_{v,B} = \frac{u^y}{\sqrt{1 - (u^x)^2/c^2}} \ . \tag{6.11}$$

These velocity components turn out to be nicely related to one another. Notice that

$$u_{v,A} = -u_{v,B}\left(\frac{1 - (u^x)^2/c^2}{1 + (u^x)^2/c^2}\right) \ . \tag{6.12}$$

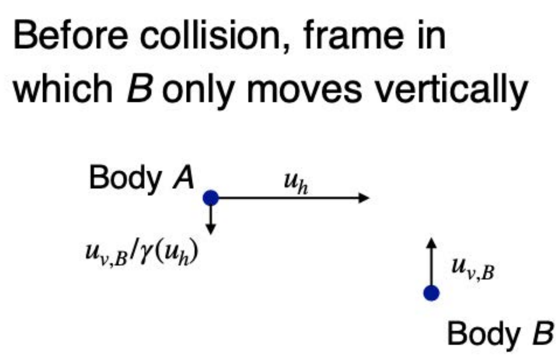
The factor in parentheses in Eq. (6.12) turns out be related to u_h in an interesting way. Calculate the value of γ for $v = u_h$:

$$\begin{aligned}
\gamma(u_h) = \frac{1}{\sqrt{1-(u_h)^2/c^2}} &= \left(1 - \frac{4(u^x)^2/c^2}{(1+(u^x)^2/c^2)^2}\right)^{-1/2} \\
&= \left(\frac{1-2(u^x)^2/c^2+(u^x)^4/c^4}{1+2(u^x)^2/c^2+(u^x)^4/c^4}\right)^{-1/2} \\
&= \left(\frac{(1-(u^x)^2/c^2)^2}{(1+(u^x)^2/c^2)^2}\right)^{-1/2} \\
&= \frac{1+(u^x)^2/c^2}{1-(u^x)^2/c^2} \, . \qquad (6.13)
\end{aligned}$$

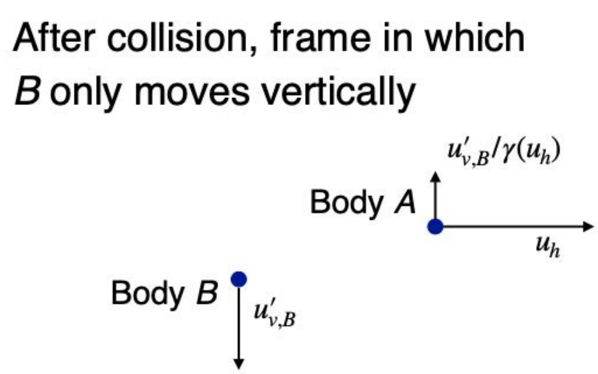
Modulo a reciprocal, this is exactly the parentheses factor in (6.12). This allows us to rewrite this equation as

$$u_{v,A} = -u_{v,B}/\gamma(u_h) \, . \qquad (6.14)$$

Let's take advantage of this to remake the figure of the collision in this frame using only the velocity components u_h and $u_{v,B}$ for our labels:



If momentum is conserved, then we expect the situation after the collision to look as follows:



The logic by which we have sketched this is that the horizontal components of the bodies' motion cannot be affected by the collision, so body A continues moving to the right with speed u_h, and body B continues to have no horizontal motion. The vertical motions reverse in direction. We leave open the possibility that the speeds associated with the vertical motion might be affected (hence the primes: $u'_{v,B}$ might differ from $u_{v,B}$).

We now demand conservation of momentum according to our hypothesized new form: both components of $\mathbf{p} = \alpha(u)m\mathbf{u}$ must be the same before and after the collision. First look at the horizontal component, for which the only contribution comes from body A:

$$\alpha\left(\sqrt{(u_h)^2 + (u_{v,B}/\gamma(u_h))^2}\right) m u_h = \alpha\left(\sqrt{(u_h)^2 + (u'_{v,B}/\gamma(u_h))^2}\right) m u_h \ . \tag{6.15}$$

The only way that this equation can hold independent of the function $\alpha(u)$ (whose nature we don't yet know) is if $u'_{v,B} = u_{v,B}$. The speed associated with the vertical components' of the bodies' velocities must be the same before and after the collision. Those velocity components simply change direction.

Require next that the vertical components of momentum be conserved:

$$\begin{aligned}\alpha(u_{v,B})mu_{v,B} - \alpha\left(\sqrt{(u_h)^2 + (u_{v,B}/\gamma(u_h))^2}\right) m u_{v,B}/\gamma(u_h) = \\ - \alpha(u_{v,B})mu_{v,B} + \alpha\left(\sqrt{(u_h)^2 + (u_{v,B}/\gamma(u_h))^2}\right) m u_{v,B}/\gamma(u_h)\end{aligned}$$

Moving similar looking factors to the same side of the equation, dividing by a common factor of $mu_{v,B}$, and multiplying by $\gamma(u_h)$, this becomes

$$\alpha\left(\sqrt{(u_h)^2 + (u_{v,B}/\gamma(u_h))^2}\right) = \gamma(u_h)\alpha(u_{v,B}) \ . \tag{6.16}$$

To simplify this, let us require that $\alpha(0) = 1$. This requirement insures that this formula recovers the Newtonian limit, which we know is an extremely good approximation for small speeds. We then examine Eq. (6.16) in the limit $u_{v,B} \to 0$:

$$\alpha(u_h) = \gamma(u_h) \ . \tag{6.17}$$

The factor $\alpha(u)$ that we hypothesized must be included in the definition of momentum works perfectly if it is the special relativistic γ factor.

In summary, the momentum defined by

$$\mathbf{p} = \gamma(u)m\mathbf{u} \tag{6.18}$$

is conserved in a universe that respects Lorentz covariance.

6.4 Kinetic energy

In Newtonian physics, the change in kinetic energy is the work done on a body: Integrating from some initial position $\mathbf{x}_i$ to a final position $\mathbf{x}_f$, we have

$$\begin{aligned}K_f - K_i &= \int_i^f \frac{d\mathbf{p}}{dt}\cdot d\mathbf{x} = \int_i^f \frac{d}{dt}(m\mathbf{u})\cdot\mathbf{u}\,dt = m\int_i^f \mathbf{u}\cdot d\mathbf{u} \\ &= \frac{1}{2}m\left(u_f^2 - u_i^2\right) \ .\end{aligned} \tag{6.19}$$

We now define relativistic kinetic energy in exactly the same way, but replace the Newtonian formula for momentum with the version we just derived:

$$\begin{aligned} K_f - K_i &= \int_i^f \frac{d\mathbf{p}}{dt} \cdot d\mathbf{x} = \int_i^f \frac{d}{dt}\left[\gamma(u) m\mathbf{u}\right] \cdot \mathbf{u}\, dt \\ &= m \int_i^f \mathbf{u} \cdot d\left[\frac{\mathbf{u}}{\sqrt{1-u^2/c^2}}\right] . \end{aligned} \tag{6.20}$$

The fact that this second form of $K_f - K_i$ is identical to the first one is not obvious. It is not difficult however to demonstrate that the two lines of (6.20) are equivalent by using the chain rule to expand the two differentials.

The final integrand that we have derived can be manipulated further:

$$\mathbf{u} \cdot d\left[\frac{\mathbf{u}}{\sqrt{1-u^2/c^2}}\right] = d\left[\frac{u^2}{\sqrt{1-u^2/c^2}}\right] - \frac{\mathbf{u} \cdot d\mathbf{u}}{\sqrt{1-u^2/c^2}} . \tag{6.21}$$

This is a very nice form: the first term on the right-hand side of (6.21) is a perfect differential; the second term on the right-hand side is simple to integrate up. Doing so, we find

$$\begin{aligned} K_f - K_i &= \left.\frac{mu^2}{\sqrt{1-u^2/c^2}}\right|_i^f - m\int_i^f \frac{\mathbf{u} \cdot d\mathbf{u}}{\sqrt{1-u^2/c^2}} \\ &= \left.\frac{mu^2}{\sqrt{1-u^2/c^2}}\right|_i^f + \left. mc^2\sqrt{1-u^2/c^2}\right|_i^f . \end{aligned} \tag{6.22}$$

For our final simplification, let's take the initial velocity to be $\mathbf{u}_i = 0$, and define $\mathbf{u}_f \equiv \mathbf{u}$. Since the initial velocity is zero, the initial kinetic energy $K_i = 0$. We set $K_f \equiv K$ and finally obtain for the kinetic energy of the system

$$\begin{aligned} K &= \frac{mu^2}{\sqrt{1-u^2/c^2}} + mc^2\sqrt{1-u^2/c^2} - mc^2 \\ &= \frac{mu^2 + mc^2 - mu^2}{\sqrt{1-u^2/c^2}} - mc^2 \\ &= \frac{mc^2}{\sqrt{1-u^2/c^2}} - mc^2 \\ &= \left[\gamma(u) - 1\right] mc^2 . \end{aligned} \tag{6.23}$$

To interpret this quantity, we define the body to have a *total* energy

$$E = \gamma(u) mc^2 ; \tag{6.24}$$

then, $E = K + mc^2$, and we interpret mc^2 as the body's **rest energy**: energy which the body possesses even when it is not in motion.

It's fair to say that Eq. (6.24) with $\gamma = 1$ is the most famous physics equation in the world. It is really interesting to pause and reflect on how it arose: we began by exploring the consequences of the hypothesis that light travels at speed c for all observers. This forced us to replace the Galilean transformation with the Lorentz transformation. This in turn mandated an adjustment to the definition of momentum. The formula $E = mc^2$, which some would argue literally changed the world, thus arose fundamentally as a consequence of this deceptively simple hypothesis.

6.5 Aside: "Relativistic mass" and why we generally don't use it anymore

In some older texts, you will see the energy and momentum defined as follows:

$$E = m(u)c^2 \;, \qquad \mathbf{p} = m(u)\mathbf{u} \;, \tag{6.25}$$

where they have defined $m(u) = \gamma(u)m$, the "relativistic mass" of the body whose "rest mass" is m. This definition rarely appears in modern relativity texts. Instead, the only "mass" used to define a body is its rest mass. The main reason for this is that m is an *invariant* — different observers assign a different energy to the body, depending on its speed u in their rest frame, but they all agree that the body's mass is m (and its energy is mc^2) in its own rest frame. As we will see in the next lecture, this invariant plays a particularly important rule in helping us to define a 4-vector which will prove to be extremely useful in helping us to keep track of energy and momentum in relativistic physics.