MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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LECTURE 8
USING 4-MOMENTUM

8.1 Introduction; a note on notation

In this lecture, we are going to examine how we use 4-momentum, seeing how it serves as a tool that combines the familiar notions of mass, momentum, and energy conservation into a single mathematical "device." One of the goals of this examination will be to see how we can use invariants to write certain quantities in ways that make the analysis easy.

Be aware that we are going overload a bit of notation, the dot product. When we write the dot product between two 4-vectors, that tells us to compute the invariant scalar product: $\vec{a} \cdot \vec{b} = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$. When we write the dot product between two 3-vectors, that's the dot product you've learned in previous physics courses: $\mathbf{a} \cdot \mathbf{b} = a^x b^x + a^y b^y + a^z b^z$. This is arguably a bit sloppy, but our use will be unambiguous as long as we consistently use the dot product symbol only in these two circumstances exactly as defined here.

Several of the examples used in this lecture are inspired by or taken from the textbook *Introduction to Elementary Particles*, by David J. Griffiths (Chapter 3).

8.2 The energy measured by a particular observer

Suppose that body A moves through spacetime with 4-momentum $\vec{p}_A$. Suppose that observer $\mathcal{O}$ has 4-velocity $\vec{u}_{\mathcal{O}}$; in our lab L we measure the components of $\vec{u}_{\mathcal{O}}$ to be ($u^t_{L,\mathcal{O}}$, $u^x_{L,\mathcal{O}}$, $u^y_{L,\mathcal{O}}$, $u^z_{L,\mathcal{O}}$). What does $\mathcal{O}$ measure for the energy of body A?

Perhaps the most straightforward way to do this would be as follows:

1. Perform a Lorentz transformation to take us to the rest frame of $\mathcal{O}$. In this frame, the components of $\vec{u}_{\mathcal{O}}$ are given by ($u^t_{\mathcal{O}}$, $u^x_{\mathcal{O}}$,$u^y_{\mathcal{O}}$,$u^z_{\mathcal{O}}$) = (c, 0, 0, 0).

2. Apply this Lorentz transformation to the components of $\vec{p}_A$; call these components $p^\alpha_{A;\mathcal{O}}$.

3. After applying the Lorentz transformation, the timelike component $p^t_{A;\mathcal{O}}$ is equal to the energy of body A in the rest frame of $\mathcal{O}$, modulo a factor of c. In other words, the energy of body A as measured by $\mathcal{O}$ is

$$E_{A;\mathcal{O}} = c\, p^t_{A;\mathcal{O}} \,. \tag{8.1}$$

This way of doing things is straightforward, and in principle we could do this to determine the energy of body A for any observer. However, note that the final result can be written

$$E_{A;\mathcal{O}} = p^t_{A;\mathcal{O}} u^t_{\mathcal{O}} - p^x_{A;\mathcal{O}} u^x_{\mathcal{O}} - p^y_{A;\mathcal{O}} u^y_{\mathcal{O}} - p^z_{A;\mathcal{O}} u^z_{\mathcal{O}} \,. \tag{8.2}$$

On initial glance, this may seem like a rather stupid way of rewriting Eq. (8.1): The term with the t components on the right-hand side of (8.2) is the same as what's on the right-hand side of (8.1), but the other 3 terms we subtract off are all equal to zero since $u_{\mathcal{O}}^{x,y,z} = 0$. However, this rewriting makes it clear that $E_{a;\mathcal{O}}$ is just the scalar product of body A's 4-momentum with observer $\mathcal{O}$'s 4-velocity, modulo a minus sign:

$$E_{A;\mathcal{O}} = -\vec{p}_A \cdot \vec{u}_{\mathcal{O}} \; . \tag{8.3}$$

This is a particularly lovely way of writing this quantity because the *scalar product is an invariant.* As long as we know the components of both $\vec{p}_A$ and $\vec{u}_{\mathcal{O}}$ in some frame of reference, we can use Eq. (8.3) to compute body A's energy as measured by $\mathcal{O}$ *without needing to perform the Lorentz transformation to the rest frame of* $\mathcal{O}$.

In addition to being a very useful way of writing the energy that some specified observer measures (we will find this form of the energy to be useful for several applications over the course of this semester), Eq. (8.3) serves as an exemplar of the power of writing things in terms of Lorentz invariants. Many times, it might be conceptually straightforward (but perhaps algebraically tedious) to figure out a quantity in a particular frame. If you can take that result and reformulate it as a Lorentz invariant, you will have a result that is broadly applicable and often much easier to apply.

Note: you might be confused about the fact that the "energy" defined by Eq. (8.3) is a Lorentz scalar. In the previous lecture, we quite specifically used energy as an example of a quantity that is **not** a scalar in relativistic physics! Are we not contradicting ourselves?

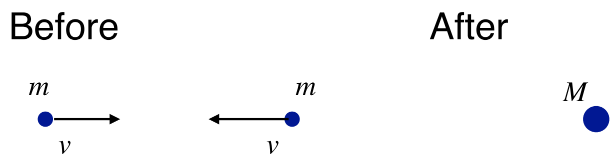
The issue here is that we are using the word "energy" for two different, albeit related, physical quantities: the timelike component of a body's 4-momentum, and the property of a body as measured by some particular observer. The first quantity we call "energy" is certainly not a Lorentz invariant — different frames assign different values to the timelike component of $\vec{p}$. The second such quantity is a Lorentz invariant because *all IRFs agree that this is the energy measured by that observer.* It is similar to the fact that the proper time experienced by some observer is a Lorentz invariant, even though "time" is certainly not Lorentz invariant. An observer's proper time may not be the time that I measure, it may not be the time that you measure, but it *is* the time which that observer measures. We all agree on that.

As a higher-level side issue, it's worth noting that a lot of confusion about various concepts in physics can be traced back to the fact that the terms we use in human language to describe things often has some built-in ambiguity. The mathematical language that we use to describe physics does not. The philosophy of "shut up and calculate," though a tad rude, is often a really way useful to get out of a confusing jam.

8.3 Collisions and decays

8.3.1 A simple collision

Let's begin by looking at some situations in which we can use conservation of 4-momentum to deduce what is going on. Begin by imagining that we smash together two lumps of clay. Each lump has mass m; we shoot them at each other, one with velocity $\mathbf{v} = v\mathbf{e}_x$, the other with $\mathbf{v} = -v\mathbf{e}_x$. They combine into a new lump of mass M. What is M?



The 4-momentum of the two lumps before we shoot them together has components

$$p^\alpha_{B,j} \doteq \begin{pmatrix} \gamma mc \\ \pm\gamma mv \\ 0 \\ 0 \end{pmatrix} , \qquad \gamma = 1/\sqrt{1 - v^2/c^2} \, . \tag{8.4}$$

The symbol "$\doteq$" we have introduced here means "the components on the left-hand side are given by the column vector on the right-hand side." Put $j = R$ to label the lump moving to the right (for which we choose the $+$ sign), and $j = L$ for the lump moving to the left (for which we choose the $-$ sign).

After the collision, we have

$$p^\alpha_A \doteq \begin{pmatrix} Mc \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{8.5}$$

The final lump is at rest in the frame we are using, so $\gamma = 1$ afterwards, and there are no non-zero spatial components to $\vec{p}_A$. Enforcing $\vec{p}_{B,R} + \vec{p}_{B,L} = \vec{p}_A$ tells us

$$M = 2\gamma(v)m \, . \tag{8.6}$$

The Newtonian expectation of course is that mass is simply conserved: $M = 2m$ in Newtonian physics. In relativity, we see that $M > 2m$. Indeed, if v is large, the amount beyond the Newtonian expectation can be significant. For instance, if $v = 3c/5$, then $M = 2.5m$ — the rest mass has increased by 25% in this case.

Where has that "extra" rest mass come from? This is $E = mc^2$ in action: kinetic energy has been converted into rest mass. When we collide two lumps at high speed, the remnant of the collision will be hotter than if we combine them at low speed. That kinetic energy gets incorporated into the random, thermal motion of the molecules that constitute the lumps. In essence, this tells us that a body's rest mass is higher when it is hot than when it is cold.

8.3.2 A simple decay

The previous example is somewhat contrived. However, it is the *time reverse* of processes that happens all the time: the *decay* of bodies with mass M_B into products whose total mass

M_A is less than the starting mass. Let's consider such a decay process: A body of mass M decays into two bodies of mass m, which then recoil in opposite directions with speed v. What is v?

Conserving 4-momentum leads us to exactly the same equation as before:

$$M = 2\gamma(v)m \; . \tag{8.7}$$

Now, we take M and m as knowns, and solve for v:

$$v = c\sqrt{1 - \left(\frac{2m}{M}\right)^2} \; . \tag{8.8}$$

Notice that if $m = M/2$, $v = 0$: all of the original rest energy turned into rest energy in the new bodies. If $m < M/2$, then some of that rest energy has become kinetic energy. (If $m > M/2$, then we've got nonsense! Check your measurements.)

8.3.3 A not-quite-so-simple decay

Although the above decay example is illustrative, it is also somewhat contrived. A more realistic example is decay into two *unequal* mass bodies. In fact, quite a few important examples involve decay into products with $m = 0$. Here's a fairly simple example: the decay of a charged pion into a muon and a massless neutrino[1]:

$$\pi^- \to \mu^- + \bar{\nu} \; . \tag{8.9}$$

This equation means that the (negatively charged) pion decays into a (negatively charged) muon and an antineutrino. This equation guarantees that charge, spin, and a quantity called "lepton number" are also conserved. If the details of this interest you, you should investigate future coursework in nuclear and particle physics. Our focus here is solely on the issue of 4-momentum conservation. We take the pion that starts this process to be at rest in our laboratory, so its 4-momentum components in the lab are given by

$$p^\alpha_\pi \doteq \begin{pmatrix} m_\pi c \\ 0 \\ 0 \\ 0 \end{pmatrix} \; . \tag{8.10}$$

[1]We now know that the neutrino has a non-zero mass, so the analysis I am presenting here is not quite right. However, the mass is so small that we have not yet actually measured it (although we have "upper bounds" on how big it can be). You should treat the idea of a massless neutrino as a very useful approximation. Hopefully we ("we" meaning the scientific community at large) will be able to refine these analyses with a mass estimate before too long.

The neutrino *cannot* be at rest: as a zero-mass particle[2] it must have non-zero 3-momentum. Let's define the neutrino's momentum as along the positive x axis:

$$p_\nu^\alpha \doteq \begin{pmatrix} E_\nu/c \\ E_\nu/c \\ 0 \\ 0 \end{pmatrix} . \tag{8.11}$$

This form of p_ν^α guarantees that $\vec{p}_\nu \cdot \vec{p}_\nu = 0$, the correct value of this invariant for a massless particle. The final quantity we need is the 4-momentum of the muon[3]. A little thought tells us that it must have the form

$$p_{\rm muon}^\alpha \doteq \begin{pmatrix} \gamma(v) m_{\rm muon} c \\ -\gamma(v) m_{\rm muon} v \\ 0 \\ 0 \end{pmatrix} . \tag{8.12}$$

This means the muon, with rest mass $m_{\rm muon}$, moves in the $-x$ direction with speed v.

Let's now enforce conservation of 4-momentum and determine (a) the energy of the neutrino, and (b) the speed v with which the pion recoils. We require both components of 4-momentum to balance:

$$\begin{aligned} & \vec{p}_\pi = \vec{p}_{\rm muon} + \vec{p}_\nu \\ t \text{ component:} \quad & m_\pi c = E_\nu/c + \gamma(v) m_{\rm muon} c \\ x \text{ component:} \quad & 0 = E_\nu/c - \gamma(v) m_{\rm muon} v . \end{aligned} \tag{8.13}$$

The x component equation allows us to eliminate E_ν from the t component equation. Doing so, we have

$$m_\pi c = \gamma(v) m_{\rm muon} (v + c) . \tag{8.14}$$

Square both sides of this and divide by c^2:

$$m_\pi^2 = m_{\rm muon}^2 \left[\frac{(v+c)^2}{c^2 - v^2}\right] = m_{\rm muon}^2 \left[\frac{c+v}{c-v}\right] . \tag{8.15}$$

Solving this for v, we find

$$v = c \left(\frac{m_\pi^2 - m_{\rm muon}^2}{m_\pi^2 + m_{\rm muon}^2}\right) . \tag{8.16}$$

From this, it's a straightforward exercise to return to the x component equation and solve for E_ν. The result is

$$E_\nu = \frac{1}{2} \left(\frac{m_\pi^2 - m_{\rm muon}^2}{m_\pi}\right) c^2 . \tag{8.17}$$

[2]Again, ignoring current wisdom that neutrinos actually have a very small mass.

[3]Note that there's potential for confusion here: we've written out the word "muon" rather than used the conventional symbol μ in order to avoid confusing μ with a downstairs index. The Greek alphabet gets a tad overused from time to time in physics.

8.3.4 A not-quite-so-simple decay, revisited

The calculation we just did is the most straightforward way to take conservation of 4-momentum and grind out the quantities of interest. You should be aware, though, that we can exploit the properties of 4-vectors to expedite our grinding of this algebra. Let's start with our initial statement of conservation of 4-momentum:

$$\vec{p}_\pi = \vec{p}_{\rm muon} + \vec{p}_\nu \; . \tag{8.18}$$

Let's move the neutrino's momentum to the left-hand side, then construct the invariant scalar product of each side with itself:

$$(\vec{p}_\pi - \vec{p}_\nu) \cdot (\vec{p}_\pi - \vec{p}_\nu) = \vec{p}_{\rm muon} \cdot \vec{p}_{\rm muon} \tag{8.19}$$

which expands to

$$\vec{p}_\pi \cdot \vec{p}_\pi - 2\vec{p}_\pi \cdot \vec{p}_\nu + \vec{p}_\nu \cdot \vec{p}_\nu = \vec{p}_{\rm muon} \cdot \vec{p}_{\rm muon} \; . \tag{8.20}$$

These various scalar products appearing in this equation take *extremely* simple forms:

$$\begin{aligned}
\vec{p}_\pi \cdot \vec{p}_\pi &= -m_\pi^2 c^2 \\
\vec{p}_{\rm muon} \cdot \vec{p}_{\rm muon} &= -m_{\rm muon}^2 c^2 \\
\vec{p}_\nu \cdot \vec{p}_\nu &= 0 \\
\vec{p}_\pi \cdot \vec{p}_\nu &= -p_\pi^0 p_\nu^0 + p_\pi^1 p_\nu^1 = -(m_\pi c)(E_\nu/c) + 0 = -m_\pi E_\nu \; .
\end{aligned} \tag{8.21}$$

Putting all these together, we have

$$m_\pi^2 - 2m_\pi E_\nu/c^2 = m_{\rm muon}^2 \tag{8.22}$$

or

$$E_\nu = \frac{1}{2}\left(\frac{m_\pi^2 - m_{\rm muon}^2}{m_\pi}\right) c^2 \; . \tag{8.23}$$

This is exactly the result for the neutrino energy we derived before.

Let's carry the analysis a few more steps in order to see a few more useful tricks. The neutrino's 3-momentum has magnitude E_ν/c, and is in the $+x$ direction. From this we know that the muon's 3-momentum has magnitude

$$|\mathbf{p}_{\rm muon}| = E_\nu/c \; , \tag{8.24}$$

and is in the $-x$ direction. Given a body's 3-momentum and mass, we can use the 4-momentum invariant to compute its energy:

$$E_{\rm muon}^2 = |\mathbf{p}_{\rm muon}|^2 c^2 + m_{\rm muon}^2 c^4 \; . \tag{8.25}$$

However, we also know that for any body

$$E = \gamma(v) m c^2 \; , \qquad \mathbf{p} = \gamma(v) m \mathbf{v} \; , \tag{8.26}$$

where $\mathbf{v}$ is that body's 3-velocity. This tells us that if we know a body's relativistic energy and relativistic momentum, we can construct its 3-velocity:

$$\mathbf{v} = \frac{\mathbf{p}c^2}{E} \; . \tag{8.27}$$

Plugging in the quantities we just found for describing pion decay, let's check the recoil velocity of the muon:

$$
\begin{aligned}
|\mathbf{v}| \equiv v &= \frac{|\mathbf{p}_{\rm muon}|c^2}{E_{\rm muon}} \\
&= \frac{E_\nu c}{\sqrt{E_\nu^2 + m_{\rm muon}^2 c^4}} \\
&= c\left(\frac{m_\pi^2 - m_{\rm muon}^2}{m_\pi^2 + m_{\rm muon}^2}\right) \; .
\end{aligned} \tag{8.28}
$$

On the last line, I plugged in our result for E_ν, and ground through a bit of algebra.

Taking advantage of the invariant scalar product often offers a quick route to isolating and finding quantities of interest in your problem. It's not a "Get Out of Algebra Free" card, but it often significantly simplifies a step or three of your analysis.

8.3.5 The center of momentum (COM) frame

Some problems can be greatly simplified by changing the reference frame in which we do the calculation. A frame that often turns out to be useful is the *center of momentum*, or COM, frame: the frame in which the total 3-momentum of the system is zero. As it happens, this has been the case in all the examples we've examined so far. This is because it just happened that these examples considered problems in which the system had no net 3-momentum in the "lab" frame in which we formulated the analysis. That is not always the case.

A classic example is a collision onto a stationary target. An important example (which I have taken from the textbook by Griffiths) is the collision of a high-speed proton with a proton which is at rest in the lab frame. One of the early experiments used this set-up to produce antiprotons, the antimatter version of protons:

$$
p + p \to p + p + p + \bar{p} \; . \tag{8.29}
$$

(Overbar on a particle means the antiparticle.) In the lab frame, here's the situation:

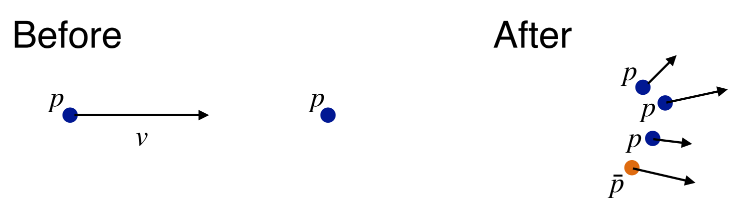


Figure 1: Proton incident on a stationary proton target; lab frame.

Our interest is to compute the *threshold* energy the incoming proton must have in order for the reaction (8.29) to occur. In the lab frame, this is hard to figure out, largely because all of the reaction products must zoom to the right in order for momentum to be conserved. But, there exists some frame in which the incident proton moves to the right (slower than in the lab frame) and the target proton moves to the left at exactly the same speed as the incident proton. The system has zero 3-momentum in that frame; by conservation, the reaction products will have total 3-momentum summing to zero as well:

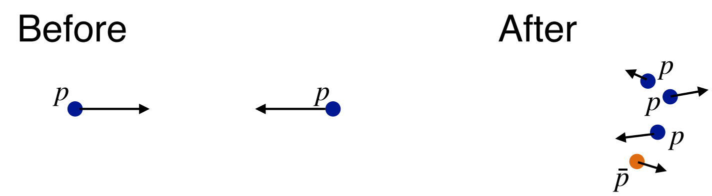


Figure 2: Proton incident on a proton target; center of momentum frame.

The "threshold" incident energy is the minimum energy necessary in order for the reaction (8.29) to proceed. With a little thought, the meaning of this energy in the COM frame should be clear: it's the energy at which the reaction products are produced with no kinetic energy. We produce only rest mass, not "wasting" any of the energy into the particles' motion (at least in this frame; they will certainly be in motion in the lab frame, since the system has net momentum in that frame).

Conservation of 4-momentum tells us that this system is governed by

$$\vec{p}_{\rm inc} + \vec{p}_{\rm target} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_{\rm anti} \; . \tag{8.30}$$

It is really easy to write down the components of the left-hand side in the lab frame:

$$p^\alpha_{\rm inc,lab} \doteq \begin{pmatrix} E_{\rm inc}/c \\ p^x \\ 0 \\ 0 \end{pmatrix} \; , \qquad p^\alpha_{\rm target,lab} \doteq \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix} \; . \tag{8.31}$$

It is also really easy to write down the components of quantities on the right-hand side at threshold in the COM frame:

$$p^\alpha_{\rm 1,COM} = p^\alpha_{\rm 2,COM} = p^\alpha_{\rm 3,COM} = p^\alpha_{\rm anti,COM} \doteq \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix} \; . \tag{8.32}$$

In these expressions, m is the proton mass, which is identical to the mass of the antiproton. (Note that in the lab frame, the 3-momentum component p^x can be rewritten using $E_{\rm inc}$ and m. Hold that thought for a moment.)

The difficulty we now face is that if we try to enforce Eq. (8.30) with what we've got so far, we're in trouble: the left-hand side and the right-hand side are expressed in different frames. The 4-momenta will not equate until we put them in the same frame. However, the *invariants* we can construct from them must equate no matter what frame we use to write down the various $\vec{p}$s. So, instead of examining Eq. (8.30), examine

$$(\vec{p}_{\rm inc} + \vec{p}_{\rm target}) \cdot (\vec{p}_{\rm inc} + \vec{p}_{\rm target}) = (\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_{\rm anti}) \cdot (\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_{\rm anti}) \; , \tag{8.33}$$

We evaluate the invariant on the left-hand side by using the components we've written down in the lab frame:

$$(\vec{p}_{\rm inc} + \vec{p}_{\rm target}) \cdot (\vec{p}_{\rm inc} + \vec{p}_{\rm target}) = -(E_{\rm inc}/c + mc)^2 + (p^x)^2 \; . \tag{8.34}$$

We evaluate the invariant on the right-hand side by using the components we've written down in the COM frame:

$$(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_{\rm anti}) \cdot (\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_{\rm anti}) = -(4mc)^2 \; . \tag{8.35}$$

Equate these two expressions, use $E^2 = |\mathbf{p}|^2c^2 + m^2c^4$ to eliminate the lab frame p^x, solve for $E_{\rm inc}$. The result is

$$E_{\rm inc} = 7mc^2 \; . \tag{8.36}$$

This means that this reaction will proceed if the incident proton's *kinetic* energy ($E_{\rm inc}$ is its *total* energy, which includes rest energy mc^2) is 6 times the proton's rest energy.

8.4 Scattering

A special case of a collision are scattering interactions: particle A comes in, interacts with particle B, and both then emerge from the interaction with new 4-momenta. Or, there could be numerous particles A_1, A_2, ... which interact with numerous particles B_1, B_2, ... This is exactly the situation we examined when we considered how to refine the definition of momentum to insure that momentum was still conserved after learning how to add velocities properly. In all cases, we are simply governed by the rule that the total 4-momentum before must equal the total 4-momentum after.

One example of a scattering interaction is particularly interesting: light interacting with a charge q of mass m_q. (The value of the charge will play no role in the calculation we are about to do, but light does not interact with non-charged bodies.) Experiments first performed by Arthur Compton in 1923 showed that in such interactions, the interaction of light with the charged body behaved just like inelastic collisions between particles. Such experiments played a large role in making it clear that light has a particle-like nature, which we call the "photon."

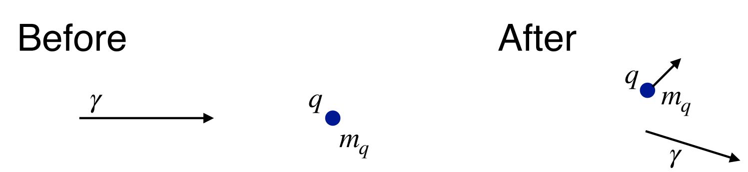


Figure 3: Compton scattering off of a charge q of mass m_q.

Suppose that the incident photon (denoted γ) comes down the x axis, but that the scattered photon makes an angle θ to the x axis. Then the situation afterwards introduces momentum along a new axis: the total momentum before they scatter is

$$p_B^\alpha \doteq \begin{pmatrix} E_\gamma/c + m_q c \\ E_\gamma/c \\ 0 \\ 0 \end{pmatrix} \; ; \tag{8.37}$$

the total momentum after scattering is

$$p_A^\alpha \doteq \begin{pmatrix} E'_\gamma/c + E_q/c \\ p_q^x + E'_\gamma \cos\theta/c \\ p_q^y - E'_\gamma \sin\theta/c \\ 0 \end{pmatrix} \; , \tag{8.38}$$

where E'_γ is the photon energy after scattering.

To make progress, we use the fact that the energy of a photon is simply related to its frequency or wavelength via

$$E_\gamma = h\nu = hc/\lambda \tag{8.39}$$

where h is *Planck's constant*:

$$h = 6.626 \times 10^{-34}\,\mathrm{J\,sec}\ . \tag{8.40}$$

Enforcing $\vec{p}_A = \vec{p}_B$ and making judicious use of our invariants, a few lines of algebra yields the Compton scattering law:

$$\lambda' = \lambda + \frac{h}{m_q c}\left(1 - \cos\theta\right)\ . \tag{8.41}$$

Some of the light's energy and momentum is transferred to the charged mass; the light is less energetic (longer wavelength) as a consequence. We will step you through this analysis on a problem set (some of you may have already seen this in quantum mechanics class).

Note that the quantity $h/m_q c$ has the dimensions of length; it is sometimes called the "Compton wavelength" of the mass m_q.

8.5 Doppler effect and aberration

The invariance of the speed of light to all observers has been the central organizing principle of almost everything we've done since Lecture 3. But this raises an interesting question: if two different frames both see a beam of light moving with speed c, what about that beam appears different to the two observers?

Let's make this concrete by examining a beam of light as seen by two observers: our station-frame observer S, and an observer T riding through the station on a train with velocity $\mathbf{v} = v\mathbf{e}_x$. Let's say that the station-frame observer reports the beam to have energy $E = h\nu$, and that it is moving in the (x, y) plane, making an angle θ with the x axis. This means that the station-frame observer measures the beam to have 4-momentum components

$$p^\alpha_S \doteq \begin{pmatrix} h\nu/c \\ h\nu\cos\theta/c \\ h\nu\sin\theta/c \\ 0 \end{pmatrix}\ . \tag{8.42}$$

What components does the observer on the train report? As usual, we apply the Lorentz transformation: $p^{\mu'}_T = \Lambda^{\mu'}{}_\alpha p^\alpha_S$, where $\Lambda^{\mu'}{}_\alpha$ is the matrix which takes events from frame S to frame T. The result is

$$p^{\mu'}_T \doteq \begin{pmatrix} \gamma h\nu/c(1 - v\cos\theta/c) \\ \gamma h\nu/c(\cos\theta - v/c) \\ h\nu\sin\theta/c \\ 0 \end{pmatrix} = \begin{pmatrix} h\nu'/c \\ h\nu'\cos\theta'/c \\ h\nu'\sin\theta'/c \\ 0 \end{pmatrix}\ . \tag{8.43}$$

The result is that, according to the train observer, the beam of light has a different energy $h\nu'$ and travels at a different angle θ'. (It's a straightforward exercise to equate the two ways I have written the components $p^{\mu'}_T$ to work out ν' and θ'.) The shift of the light's energy is the Doppler effect, the same basic physics by which we hear the frequency of a siren change pitch as an emergency vehicle drives past us at high speed. The change in angle is aberration. You explored the phenomenon of light's trajectory changing angle according to different observers on a recent problem set; such an analysis can be done quite elegantly using 4-momentum.