Massachusetts Institute of Technology
Department of Physics
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Lecture 13
Accelerations and forces

13.1 An apparent paradox

Consider a pair of twins. Twin A stays on Earth. Twin B travels on a rocket ship to Alpha Centauri, 4 light years away, moving at 99% of the speed of light. Twin B then turns around and comes back. When they get together, which one is older?

The essence of the apparent paradox is that, according to special relativity, no inertial observer is preferred:

- Twin A says that B is in motion. Therefore, B's clock runs slow, and B is younger.
- Twin B says that A is in motion. Therefore, A's clock runs slow, and A is younger.

When the twins reunite, they can't both be right — one of them has unambiguously aged more than the other. Who has used the wrong logic?

Twin B has used the wrong logic, because they forgot that **they are not an inertial observer**. Twin B accelerates (3 times: once from Earth to start the trip, once at Alpha Centauri to turn around and come back, and once upon returning to Earth). This acceleration breaks the symmetry between the twins.

Does this mean that Twin B is older or younger? To answer this, we need to think about accelerated motion.

13.2 4-acceleration; the Momentarily Comoving Reference Frame (MCRF)

We begin by quickly re-examining the notion of 4-acceleration, which was briefly introduced in our discussion of the Lorentz force. We define the 4-acceleration by

$$\vec{a} = \frac{d\vec{u}}{d\tau} \,, \tag{13.1}$$

i.e., the rate of change of 4-velocity per unit proper time. As discussed in that earlier lecture, its invariant scalar product with $\vec{u}$ is zero, which follows from $\vec{u}\cdot\vec{u} = -c^2$:

$$\frac{d}{d\tau}\left(\vec{u}\cdot\vec{u}\right) = \vec{a}\cdot\vec{u} + \vec{u}\cdot\vec{a} = 2\vec{a}\cdot\vec{u} = 0 \,. \tag{13.2}$$

This is in sharp contrast to the 3-acceleration $\mathbf{a}$, since physics imposes no constraints on the value of $\mathbf{a}\cdot\mathbf{u}$ (here using the "old-fashioned" dot product between two 3-vectors).

To wrap our heads around the physics of acceleration, let's introduce a particular special reference frame: the MCRF, or *Momentarily Comoving Reference Frame*. The MCRF is a Lorentz frame that, at least for one moment, has the same velocity as the accelerating observer. An accelerating observer is at rest in the MCRF for one moment.

In the MCRF, the following properties describe the motion of the accelerating observer:

$$u^t_{\rm MCRF} = c\ , \qquad u^{x,y,z}_{\rm MCRF} = 0\ , \tag{13.3}$$

$$d\tau = dt_{\rm MCRF}\ . \tag{13.4}$$

These properties tell us that

$$a^\mu_{\rm MCRF} \doteq \begin{pmatrix} 0 \\ du^x_{\rm MCRF}/dt \\ du^y_{\rm MCRF}/dt \\ du^z_{\rm MCRF}/dt \end{pmatrix}\ . \tag{13.5}$$

This form guarantees that $\vec a \cdot \vec u = 0$: if you evaluate that scalar product using the components defined in the MCRF, you can see quite clearly that it holds. But, the scalar product is an invariant — if it is true in one frame, then it is true in all frames.

The MCRF thus helps us to understand what 4-acceleration means. Suppose some observer has a 4-acceleration $\vec a$, and that we find $\vec a \cdot \vec a = a^2$. (Note that $\vec a$ must be spacelike[1] in order for $\vec a \cdot \vec u = 0$.) Then a represents the magnitude of the acceleration that is experienced by the accelerating observer in the MCRF. It is the acceleration that this observer feels in their own rest frame.

13.3 A uniformly accelerated observer

Let's imagine an observer who starts at rest with respect to us, but who experiences uniform acceleration with magnitude $g = 10\,{\rm m/sec}^2$. Let this acceleration be in the x direction. "Uniform" means that the observer feels this acceleration at all times, so that $\vec a \cdot \vec a = g^2$ at all times. The acceleration in the MCRF is always the same — even though the MCRF itself is continually changing as the observer accelerates. Can we compute the 4-velocity at later times?

We have two initial conditions: $\vec u(\tau = 0) = c\vec e_t$ and $\vec a(\tau = 0) = g\vec e_x$. We also have three constraints:

$$\vec u \cdot \vec u = -c^2 \qquad \text{at all times}\ , \tag{13.6}$$

$$\vec u \cdot \vec a = 0 \qquad \text{at all times}\ , \tag{13.7}$$

$$\vec a \cdot \vec a = g^2 \qquad \text{at all times}\ . \tag{13.8}$$

Let's write out these constraint equations, using the fact that $a^\mu = du^\mu/d\tau$:

$$-(u^t)^2 + (u^x)^2 = -c^2\ , \tag{13.9}$$

$$-u^t\frac{du^t}{d\tau} + u^x\frac{du^x}{d\tau} = 0\ , \tag{13.10}$$

$$-\left(\frac{du^t}{d\tau}\right)^2 + \left(\frac{du^x}{d\tau}\right)^2 = g^2\ . \tag{13.11}$$

[1]To be lightlike, we must have $\vec a \cdot \vec a = 0$. That's only true if $a = 0$, an uninteresting limit.

Staring at these equations a bit and thinking about some functions we know suggests that hyperbolic functions might be useful here. Let's try

$$u^t = c\cosh(A\tau) \ , \qquad u^x = c\sinh(A\tau) \ . \tag{13.12}$$

It's not hard to see that this form guarantees Eqs. (13.6) and (13.7) will work. Enforcing Eq. (13.8) gives us the constant A:

$$\frac{du^t}{d\tau} = cA\sinh(A\tau) \ , \qquad \frac{du^x}{d\tau} = cA\cosh(A\tau) \ ; \tag{13.13}$$

so

$$-\left(\frac{du^t}{d\tau}\right)^2 + \left(\frac{du^x}{d\tau}\right)^2 = c^2A^2\left[-\sinh^2(A\tau) + \cosh^2(A\tau)\right] = g^2 \tag{13.14}$$

which tells us that

$$A = \frac{g}{c} \ . \tag{13.15}$$

Our complete solution for the uniformly accelerated observer is thus

$$\vec{u} = c\cosh(g\tau/c)\,\vec{e}_t + c\sinh(g\tau/c)\,\vec{e}_x \ , \tag{13.16}$$

$$\vec{a} = g\sinh(g\tau/c)\,\vec{e}_t + g\cosh(g\tau/c)\,\vec{e}_x \ , \tag{13.17}$$

where τ is the proper time experienced by this observer since their trip started.

Let's use this solution to explore what happens when someone is uniformly accelerated. Two questions are at the top of our list:

1. After traveling for time T as measured *by the accelerating observer* (i.e., for a total experienced proper time $\tau = T$), how far has the observer traveled?

2. After traveling for time T as measured *by the accelerating observer*, how much time has elapsed "back home"?

Both questions are answered by integrating the 4-velocity. Let's look at how far they've traveled first:

$$\begin{aligned}\Delta x &= \int_0^T u^x\,d\tau \\ &= c\int_0^T \sinh\left(\frac{g\tau}{c}\right)d\tau \\ &= \frac{c^2}{g}\left[\cosh\left(\frac{gT}{c}\right) - 1\right] \ .\end{aligned} \tag{13.18}$$

Using the fact that $c^2/g = 0.96940$ light years, and $(g/c) = 1.0316\,\text{year}^{-1}$, we can make a table of distance versus time experienced by the accelerating observer:

- $\Delta x(T = 1\,\text{year}) = 0.56318\,\text{light year}$
- $\Delta x(T = 2\,\text{years}) = 2.9071\,\text{light years}$
- $\Delta x(T = 5\,\text{years}) = 83.268\,\text{light years}$
- $\Delta x(T = 10\,\text{years}) = 14{,}638\,\text{light years}$

How much time back in the original frame elapses while doing this?

$$\begin{aligned}
\Delta t &= \int_0^T (u^t/c)d\tau \\
&= \int_0^T \cosh\left(\frac{g\tau}{c}\right) d\tau \\
&= \frac{c}{g}\sinh\left(\frac{gT}{c}\right) .
\end{aligned} \tag{13.19}$$

The equivalent table for time elapsed reads

- $\Delta t(T = 1\,\text{year}) = 1.1870\,\text{years}$
- $\Delta t(T = 2\,\text{years}) = 3.7533\,\text{years}$
- $\Delta t(T = 5\,\text{years}) = 84.232\,\text{years}$
- $\Delta t(T = 10\,\text{years}) = 14{,}639\,\text{years}$

As seen back in the original frame, the accelerated observer is getting closer and close to the speed of light, and so is experiencing enormous time dilation. Their 10 year interval is over 14,600 years in the original frame — their moving clock is running *very* slowly compared to a clock in the original frame.

13.4 Forces

We encountered forces briefly in our discussion of electromagnetic effects. In this section, we return to this discussion, and put a few details on a more solid footing.

Two general conceptual frameworks are used:

1. We can define a 4-force, $\vec{F} = d\vec{p}/d\tau$. In terms of this, we have $\vec{a} = \vec{F}/m$. In principle, this is the way you might imagine we want to do things, since $\vec{F}$ is a spacetime 4-vector. It is straightforward for us to transform the components of $\vec{F}$ to different reference frames, so this would seem to be the ideal quantity for bringing forces into a relativistic discussion.

2. We can use the usual 3-force, $\mathbf{F} = d\mathbf{p}/dt$. This is fine, as long as we recognize that $\mathbf{p}$ and t are the momentum and time *as measured in a particular frame*, and that we must be careful when we transform them between frames. Changing frames will transform $\mathbf{F}$ in a way that is rather more complicated than a simple Lorentz transformation since quantities in both the numerator and the denominator of the force's definition are affected by this change of representation.

This being a relativity class, you might think we have a preference for the 4-force formulation. However, the 3-force is in fact quite useful and important. This is because we always perform our measurements in some particular frame, using the time and space coordinates of that frame, and pinning down the momentum and energy in that frame. So it is quite useful for us to understand how 3-forces transform between frames as well as 4-forces. Ideally, we'd like to know how to flip back and forth between the two descriptions, as both are important and useful.

Let's go back to our train and station frames. Imagine that a body has a 3-velocity $\mathbf{u}$ as measured in a station, and so has 3-momentum $\mathbf{p}_S = \gamma(u)m\mathbf{u}$ and energy $E_S = \gamma(u)mc^2$ according to the station-frame observers. A train moves through a station with velocity $\mathbf{v} = v\mathbf{e}_x$. If force $\mathbf{F}_S$ acts on the body in the station, what is the force $\mathbf{F}_T$ that acts on the body according to measurements on the train?

When in doubt, go back to the Lorentz transformation. We know that $\mathbf{F} = d\mathbf{p}/dt$, so let's examine the key quantities appearing here and how they transform between frames. Start with the x component:

$$\begin{aligned}(\mathbf{F}_T)^x = \frac{dp^x_T}{dt_T} &= \frac{\gamma\,(dp^x_S - v dE_S/c^2)}{\gamma\,(dt_S - v\,dx_S/c^2)} \\ &= \frac{(\mathbf{F}_S)^x - (v/c^2)(dE_S/dt_S)}{1 - v(\mathbf{u})^x/c^2}\ . \end{aligned} \tag{13.20}$$

Notice we have to a little careful with notation, since the letter "F" is used for both the 4-force and the 3-force and the letter "u" is used for 3-velocity in some frame and 4-velocity. The convention we are using is that F^i represents the ith component of the 4-force, but $(\mathbf{F})^i$ represents the ith component of the 3-force; u^i and $(\mathbf{u})^i$ have analogous meanings for 4-velocity and 3-velocity components, respectively.

We can simplify Eq. (13.20) a bit more. We know that $E^2 = p^2c^2 + m^2c^4$ for the body. Evaluating everything in the station frame and taking derivatives with respect to station time, we have

$$\begin{aligned} E_S\frac{dE_S}{dt_S} &= \mathbf{p}_S \cdot \frac{d\mathbf{p}}{dt_S}c^2 \\ \gamma mc^2\frac{dE_S}{dt_S} &= \gamma m\mathbf{u} \cdot \frac{d\mathbf{p}}{dt_S}c^2 \\ \longrightarrow \qquad \frac{dE_S}{dt_S} &= \mathbf{F}_S \cdot \mathbf{u}\ . \end{aligned} \tag{13.21}$$

So, we find that the x component of the force transforms as

$$(\mathbf{F}_T)^x = \frac{(\mathbf{F}_S)^x - (v/c^2)\mathbf{F}\cdot\mathbf{u}}{1 - v(\mathbf{u})^x/c^2}\ . \tag{13.22}$$

You may notice a resemblance to the velocity addition formula! Indeed, working out the other two components, we find

$$(\mathbf{F}_T)^{y,z} = \frac{(\mathbf{F}_S)^{y,z}}{\gamma(1 - v(\mathbf{u})^x/c^2)}\ . \tag{13.23}$$

Although we have spent some time (and ink/chalk) developing how the 3-force transforms between frames of reference, it should be emphasized that the 4-force is also used quite a lot. The 4-force fits more naturally into a "spacetime" language; the 3-force is more naturally suited to the "space" plus "time" language adapted to a particular observer. Some forces may be very naturally expressed using the 4-force, but we then may need the 3-vector components in order to assess what some observer will measure in their lab. It is important to develop fluency translating back and forth between these different notions of the force.

So, how do we relate these two notions of force? The analysis is somewhat similar to how we relate 4-velocity components to 3-velocity components. Let's consider the spatial components first:

$$F^i = \frac{dp^i}{d\tau} . \tag{13.24}$$

The interval $d\tau$ is as measured on the clock of the body which experiences this force. It is related to time as seen in that frame by $d\tau = dt/\gamma(u)$, where u is the magnitude of the body's 3-velocity in that frame. This means

$$F^i = \gamma(u)\frac{dp^i}{dt} = \gamma(u)(\mathbf{F})^i . \tag{13.25}$$

Next consider the timelike component:

$$F^0 = \frac{dp^0}{d\tau} = \gamma(u)\frac{d}{dt}\left(\frac{E}{c}\right) = \frac{\gamma(u)}{c}\frac{dE}{dt} . \tag{13.26}$$

We already showed that $dE/dt = \mathbf{F}\cdot\mathbf{u}$. Putting this all together, we have a "glossary" that lets us switch back and forth between the 4-vector and 3-vector notions of force:

$$F^0 = \frac{\gamma(u)}{c}\mathbf{F}\cdot\mathbf{u} , \tag{13.27}$$

$$F^i = \gamma(u)(\mathbf{F})^i . \tag{13.28}$$