Massachusetts Institute of Technology
Department of Physics
8.033 Fall 2024

Lecture 19
From weak gravity to strong gravity

19.1 A "strong gravity" spacetime

In the previous lecture, we described a few exact solutions that have been found to Einstein's field equations of general relativity, and we discussed in some detail how things behave in the spacetime that describes "weak" gravity. For a spherical body, this can be taken to be gravity for which $GM/rc^2 \ll 1$ everywhere. We found that in this spacetime:

- freely-falling bodies move in a way that reproduces the predictions of Newtonian gravity;
- clocks "lower" in the spacetime (i.e., at smaller r) tick more slowly than those at higher altitudes in a way that is exactly consistent with the redshift of light[1];
- although we skipped over many of the details, terms beyond the ones which reproduce the predictions of Newtonian gravity explain a centuries-old mystery about the precession of Mercury's orbit about the Sun;
- and finally, as you will show on problem set #9, the trajectory of light bends as it passes near a gravitating body. A celebrated measurement by Dyson and Eddington in 1919 confirmed[2] the predictions of general relativity; indeed, the publicity[3] surrounding the light-bending measurement was a huge part of what turned Albert Einstein from a highly respected scientist into an international public figure.

These items went a long way toward convincing most scientists that general relativity provides a valid relativistic theory of gravity. Most people are happy to work under the assumption that gravity is described by spacetimes which solve the equation $G^{\mu\nu} = (8\pi G/c^4)T^{\mu\nu}$.

However, as we noted in the previous lecture, this is not the *only* way to combine relativity with gravity. Indeed, as was briefly described in Lecture 18, there's a certain sense in which general relativity can be regarded as the *simplest* theory of relativistic gravity. Perhaps differences between theory and measurement will arise as we investigate strong gravity —

[1] We didn't actually look at light propagation yet; we will do that in this lecture.

[2] There has been some controversy about whether this measurement's error bars are as good as was claimed. Independent of that controversy (which has been thoroughly investigated; the consensus is that the measurement by Dyson and Eddington was valid, though it is worth digging into the details), the bending of light by gravity has been thoroughly examined many times since 1919, and general relativity's predictions hold up. Indeed, they hold up so well that these days people *assume* that general relativity correctly describes light bending, and use it to learn about the properties of large distributions of mass by measuring how light bends. This is what the astronomical science of *gravitational lensing* is all about.

[3] In no small part because an expedition by British scientists to examine what was then regarded as a German theory was treated as a welcome example of the scientific community setting aside the antagonism of World War I to focus on truths that transcend national borders.

after all, if you want to push the boundaries in physics, you take the framework in which you interpret your measurements and either figure out how to measure things with greater and greater precision, or you push into regimes beyond what you have already investigated (or both).

In this lecture, we're going to explore what general relativity tells us about when gravity is not weak — i.e., in situations where it is not the case that $GM/rc^2 \ll 1$. Our tool for this exploration is the Schwarzschild metric, for which the line element takes the form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -\left(1 - \frac{2GM}{rc^2}\right)(c\,dt)^2 + \frac{dr^2}{(1 - 2GM/rc^2)} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \; . \quad (19.1)$$

This spacetime is exact, and holds for all r. By using the full mathematical machinery of general relativity, one finds that (19.1) exactly describes a spacetime for which $T^{\mu\nu} = 0$. However, this spacetime also describes the spherically symmetric gravity of a mass M.

What this is telling us is that Eq. (19.1) describes the gravity of a mass M, but there's no matter or energy density anywhere. So, what does *that* mean? Perhaps the simplest way of understanding this (admittedly counterintuitive) aspect of the Schwarzschild solution is by analogy. If you take the Coulomb point charge electric field,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\mathbf{x} \; , \quad (19.2)$$

and apply the divergence operator to it, you get zero. This means that the charge density everywhere is zero:

$$\rho = \epsilon_0\left(\nabla \cdot \mathbf{E}\right) = 0 \; . \quad (19.3)$$

So there's no charge density ... but when we integrate it up, we get a non-zero charge q.

The resolution of this apparent paradox in electrostatics is that the divergence is actually an ill-behaved operation exactly at the origin, $\mathbf{x} = \mathbf{0}$. In courses like 8.07, we learn that we can resolve this by introducing a singular "function"[4] that essentially puts a finite amount of charge into a zero-volume point at the origin. At least heuristically, something similar is going on with the Schwarzschild spacetime — at least in classical general relativity, there's a singular point at the coordinate $r = 0$ where general relativity's equations are ill-behaved. But everywhere away from that point, there is no problem.

Thanks to non-linear terms in Einstein's field equations, the $r = 0$ singularity is even more disturbing and hard to deal with than the analogous Coulomb singularity. Nonetheless, it is useful to set aside misgivings about this spacetime and examine what it tells us. (Indeed, an aspect of the spacetime's nature we will soon investigate suggests that any "weirdness" near $r = 0$ is not of concern — at least, not of immediate concern. We will elaborate on this cryptic remark soon.)

Let us begin by again looking at an observer who is at rest in the spacetime, and think about how their clocks behave. Notice that as $r \to \infty$, the Schwarzschild metric is nothing more than the metric of special relativity (albeit in spherical coordinates — you can transform from the inertial coordinate form we've long been using by the transformations $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$). This tells us that, as in the "weak gravity" metric of the previous lecture, t describes clocks that are used by very distant observers. This means t makes a useful "bookkeeper" time for comparing different observers' clocks.

[4]Strictly-speaking, the quantity we use is not a function, but it can be treated much like a function if we are careful. If this is new to you and you are curious about this, look up the *Dirac delta function*. This is a topic for another day, and another course.

Let's compare the bookkeeper time with the time of an observer who is spatially at rest at some radius r. We put $u^r = 0$, $u^\theta = 0$, $u^\phi = 0$; invoking the principle of equivalence, we require $\vec{u} \cdot \vec{u} = -c^2$ to solve for $u^t = c\, dt/d\tau$:

$$\vec{u} \cdot \vec{u} = -\left(1 - \frac{2GM}{rc^2}\right)\left(c\frac{dt}{d\tau}\right)^2 = -c^2 \ , \tag{19.4}$$

which means

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - 2GM/rc^2}} \qquad \text{or} \qquad \Delta\tau(r) = \Delta t\sqrt{1 - \frac{2GM}{rc^2}} \ . \tag{19.5}$$

Notice that if $r \gg 2GM/rc^2$, we can use the binomial expansion and approximate:

$$\sqrt{1 - \frac{2GM}{rc^2}} \simeq 1 - \frac{GM}{rc^2} \qquad \text{for} \qquad r \gg 2GM/rc^2 \ . \tag{19.6}$$

At clock located at coordinate r ticks slower than a clock that is very far away by a factor GM/rc^2, exactly the variation in clock ticking that we found in the weak-gravity metric. This confirms that the Schwarzschild metric agrees with our previous results in the right limit. However, the rate at which clocks slow as r gets slower is *far* more extreme than what we saw in the weak-field case (remember that the weak field formula was only valid if $r \gg GM/c^2$ everywhere). Indeed, (19.5) predicts that our observer's clock *stops* as $r \to 2GM/c^2$ — and it appears to break down completely when $r < 2GM/c^2$.

So what is going on with *that*??

19.2 Light propagation

The propagation of light was one of our most important tools for making sense of how space and time behave in special relativity. Light propagation helps us in general relativity too, though we need to lay out a few rules for how we are going to use it.

We cannot define 4-velocity along a light ray — because the speed is c, proper time is not defined along it. However, 4-momentum is perfectly well defined along a light ray. Let us look at the 4-momentum of a body with mass m:

$$\vec{p} = m\frac{d\vec{x}}{d\tau} \ . \tag{19.7}$$

Let us define a parameter λ such that $d\lambda = d\tau/m$. Then,

$$\vec{p} = \frac{d\vec{x}}{d\lambda} \ . \tag{19.8}$$

If we consider a sequence of bodies with ever decreasing m, we can define the 4-momentum of light to be $\vec{p} = d\vec{x}/d\lambda$ in the limit $m \to 0$. The parameter λ can then be regarded as a kind of "tick mark" that allows us to label events along a light ray, with units chosen so that $d\vec{x}/d\lambda$ yields a quantity with the units of momentum.

Since the Schwarzschild spacetime is spherically symmetric, let's examine light rays that propagate radially, setting $p^\theta = p^\phi = 0$. The defining characteristic of a null or light-like

4-momentum in special relativity was that $\vec{p} \cdot \vec{p} = 0$. Invoking the principle of equivalence, the same thing holds in general relativity:

$$\vec{p} \cdot \vec{p} = g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = -\left(1 - \frac{2GM}{rc^2}\right) \left(c\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \left(\frac{dr}{d\lambda}\right)^2 = 0 \ . \tag{19.9}$$

Using this, we can solve for the speed at which light propagates in this coordinate system:

$$\frac{dr}{dt} = \pm c \left(1 - \frac{2GM}{rc^2}\right) \ . \tag{19.10}$$

Notice this appears to tell us that the light is not propagating at speed c! Please bear in mind, however, that this is the light's speed *in this coordinate system*. Equation (19.10) expresses the ratio of an interval of radial coordinate r to an interval of coordinate time t. Consider two events: one is at (t, r, θ, ϕ); the other is at $(t, r + dr, \theta, \phi)$. The distance between these events is given by $ds = dr/\sqrt{1 - 2GM/rc^2}$. This distance is *larger* than dr. So when the light moves through a *coordinate* distance dr, the spatial distance it moves is greater than dr. Note also that this speed is defined in terms of the time used by observers who are very far away. The clocks of observers near r tick more slowly than the clocks of distant observers. With a little effort, one can show that observers will always see light move with speed c when things are expressed as physical distance divided by their own time. The idea that the speed of light is c for all observers has not been broken; indeed, thanks to the principle of equivalence, it remains foundational to this subject.

That said, Eq. (19.10) has very interesting behavior in the limit $r \to 2GM/c^2$ — the coordinate velocity there is *zero*. That suggests that a light ray "launched" radially outward (or inward, for that matter) at $r = 2GM/c^2$ will stay there forever. This appears to contradict the principles outlined in the previous paragraph. However, recall from Eq. (19.5) that an observer's clock stops relative to a distant clock when we reach this radius. The radius $r = 2GM/c^2$ is indeed special, and a bit weird. More on this radius below.

Let's look at one more aspect of the behavior of light — its energy as it propagates outwards from some radius. Before doing this, it is very useful to pause and look at the *Lagrangian* for light propagating in the Schwarzschild spacetime. We defined $L = g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta/2$ as the Lagrangian for material bodies moving through the spacetime $g_{\alpha\beta}$ with $\dot{x}^\alpha \equiv dx^\alpha/d\tau$. By adjusting the definition so that $\dot{x}^\alpha \equiv dx^\alpha/d\lambda$, the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^\alpha} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = 0 \tag{19.11}$$

can then be used to describe light moving through the spacetime.

The Lagrangian for a light ray is given by

$$L = \frac{1}{2}\left[-\left(1 - \frac{2GM}{rc^2}\right)\left(c\dot{t}\right)^2 + \frac{(\dot{r})^2}{(1 - 2GM/rc^2)} + r^2(\dot{\theta})^2 + r^2 \sin^2\theta (\dot{\phi})^2\right] \ , \tag{19.12}$$

where $\dot{x}^0 \equiv c\dot{t} = c\, dt/d\lambda = p^t$, $\dot{r} = dr/d\lambda = p^r$, etc.

Notice that $\partial L/\partial x^0 = (1/c)\partial L/\partial t = 0$. By one of the exercises you did on problem set #8, this tells us that

$$\frac{1}{c}\frac{\partial L}{\partial \dot{t}} = -\left(1 - \frac{2GM}{rc^2}\right) c\dot{t} = \text{constant} \ . \tag{19.13}$$

Using the fact that $c\dot{t} = p^t$ and $-(1 - 2GM/rc^2) = g_{tt}$, this tells us that along the light ray

$$g_{tt}p^t \equiv p_t = \text{constant} . \tag{19.14}$$

The *downstairs* t component of the 4-momentum, p_t, is a constant along the light ray's trajectory.

Let's use this to compare the energy that is measured by a static observer at $r = R$ with an observer who is very far away, $r \to \infty$. We use the fact that the energy measured by an observer whose 4-velocity is $\vec{u}$ is given by $E_{\vec{u}} = -\vec{p} \cdot \vec{u}$ — by the equivalence principle, this result (which we developed in special relativity) will work just fine for us in general spacetimes. We use the fact that an observer who holds static at $r = R$ has a 4-velocity with components $u^t = c(dt/d\tau) = c/\sqrt{1 - 2GM/rc^2}$, $u^r = u^\theta = u^\phi = 0$. So then

$$\begin{aligned}
\frac{E(r \to \infty)}{E(r = R)} \equiv \frac{E_\infty}{E_R} &= \frac{-\vec{p} \cdot \vec{u}\big|_{r\to\infty}}{-\vec{p} \cdot \vec{u}\big|_{r=R}} \\
&= \frac{p_t u^t(r \to \infty)}{p_t u^t(r = R)} \\
&= \frac{1}{1/\sqrt{1 - 2GM/Rc^2}} \\
&= \sqrt{1 - \frac{2GM}{Rc^2}} .
\end{aligned} \tag{19.15}$$

The first line of this relation just inserts the definition $E = -\vec{p} \cdot \vec{u}$. The second line expands the inner product, using the downstairs form of the 4-momentum and the upstairs form of the 4-velocity, taking advantage of the fact that only $u^t \neq 0$. On the third line, we use the fact that p_t is a constant along the light ray's trajectory to cancel it out — p_t has the same value at $r = R$ as it does in the limit $r \to \infty$. We also use the solution for u^t that we derived earlier in this lecture.

The final line shows us how light is redshifted as propagates from $r = R$ out to infinity. Notice once again the interesting behavior as $R \to 2GM/c^2$: the light is so redshifted in this case that the energy measured very far away is *zero*. **None of the light's energy gets out if it starts at $R = 2GM/c^2$.**

To summarize, our investigation of the Schwarzschild spacetime has yielded the following outcomes:

- Clocks run slower at smaller values of r. If $d\tau_R$ is an interval of time measured at $r = R$, and dt is an interval measured by clocks very far away ($r \to \infty$), then we find

$$d\tau_R = dt\sqrt{1 - \frac{2GM}{Rc^2}} . \tag{19.16}$$

- Light that is emitted from $r = 2GM/c^2$ appears to move in the radial direction with coordinate speed $dr/dt = 0$. In other words, light does not seem to ever move away from this radius.

- If light is emitted from radius $r = R$ with energy E_R, then it is measured far away to have energy

$$E_\infty = E_R\sqrt{1 - \frac{2GM}{Rc^2}} . \tag{19.17}$$

This is consistent with the redshifting of light we saw in other contexts, but notice that $E_\infty \to 0$ as $R \to 2GM/c^2$.

This all tells us that there is something quite interesting about the radius $r = 2GM/c^2$. Let's do one more calculation, which if all goes well will really confuse us.

19.3 The trajectory of an infalling observer

Imagine an observer who starts at rest from $r = R$ and then falls. Suppose they have no motion in the θ or ϕ directions. The Lagrangian describing their motion is then given by

$$L = \frac{1}{2} g_{\alpha\beta} u^\alpha u^\beta = -\frac{1}{2}\left(1 - \frac{2GM}{rc^2}\right)\left(c\frac{dt}{d\tau}\right)^2 + \frac{1}{2}\frac{(dr/d\tau)^2}{1 - 2GM/rc^2} \ . \tag{19.18}$$

On problem set #8, you found that because $\partial L/\partial t = 0$, it must be the case that $\partial L/\partial \dot{t}$ is a constant along the body's trajectory. We call this constant the body's energy per unit mass (up to a minus sign) because of its limiting behavior as $r \to \infty$:

$$E = -\frac{\partial L}{\partial \dot{t}} = c^2\left(1 - \frac{2GM}{rc^2}\right)\frac{dt}{d\tau} = \text{constant} \ . \tag{19.19}$$

This observer starts at rest, and we know that for an observer who is at rest in the Schwarzschild spacetime

$$\frac{dt}{d\tau} = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} \ . \tag{19.20}$$

Applying this to our infalling observer when they are at rest at $r = R$, we find

$$E_{\rm obs} = c^2\sqrt{1 - \frac{2GM}{Rc^2}} \ . \tag{19.21}$$

We also know that $\vec{u}\cdot\vec{u} = -c^2$:

$$-c^2 = -\left(1 - \frac{2GM}{rc^2}\right)\left(c\frac{dt}{d\tau}\right)^2 + \frac{(dr/d\tau)^2}{1 - 2GM/rc^2} \ . \tag{19.22}$$

We can clean this up, using Eq. (19.19) to replace $dt/d\tau$ with $E_{\rm obs}$ and a function of r. After making this substitution, we can rearrange to make an equation describing the infalling observer's trajectory with respect to r:

$$\begin{aligned} \left(\frac{dr}{d\tau}\right)^2 &= \frac{E_{\rm obs}^2}{c^2} - c^2\left(1 - \frac{2GM}{rc^2}\right) \\ \longrightarrow \quad \frac{dr}{d\tau} &= -\sqrt{\frac{2GM}{r} - \frac{2GM}{R}} \ . \end{aligned} \tag{19.23}$$

The second line of Eq. (19.23) uses the value of $E_{\rm obs}$ we found above; we choose an overall minus sign for the square root to give us infall.

Equation (19.23) is most easily solving by finding $\tau(r)$ — i.e., the elapsed proper time that passes after the observer has fallen from R to r. The result is

$$\tau = \sqrt{\frac{1}{2GM}}\left[R^{3/2}\arctan\left(\sqrt{\frac{R-r}{r}}\right) + \sqrt{rR(R-r)}\right] \ . \tag{19.24}$$

This tells us that the observer falls on a rather smooth trajectory according to their own clocks, reaching $r = 0$ in finite proper time:

$$\Delta\tau(r = R \to r = 0) = \frac{\pi}{2}\sqrt{\frac{R^3}{2GM}} \,. \tag{19.25}$$

Despite the fact that $r = 2GM/c^2$ seems to be quite important, nothing special happens here as r passes through this coordinate.

Parameterizing the motion in terms of the observer's proper time is fine for discussing how they see their own motion. But how does it look to a distant observer, someone who is watching that person fall in from a safe distance? Very distant observers use the coordinate t for their clocks, and an interesting question is how the motion looks when parameterized in a way that suits their perspective. We know that the infalling observer's clocks "run slow" according to distant observers. We thus expect that a process which happens quickly according to the infalling observer's clock may not look quite so fast as seen by someone very far away.

We begin by working out the infall as parameterized by t:

$$\begin{aligned}\frac{dr}{dt} &= \frac{dr}{d\tau}\left(\frac{dt}{d\tau}\right)^{-1}\\ &= -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{R}\right)}\frac{c^2}{E_{\rm obs}}\left(1 - \frac{2GM}{rc^2}\right) \,.\end{aligned} \tag{19.26}$$

Using $E_{\rm obs} = c^2\sqrt{1 - 2GM/Rc^2}$, we can solve this for $t(r)$. The solution is the rather more complicated expression

$$\begin{aligned}t(r) = {}& \frac{2GM}{c^3}\ln\left[\frac{\sqrt{\frac{r(R-2GM/c^2)}{2GM(R-r)/c^2}} + 1}{\sqrt{\frac{r(R-2GM/c^2)}{2GM(R-r)/c^2}} - 1}\right] + \sqrt{r(R-r)\left(\frac{Rc^2}{2GM} - 1\right)}\\ &+ \left(R + \frac{4GM}{c^2}\right)\sqrt{\frac{Rc^2}{2GM} - 1}\left[\frac{\pi}{2} - \arctan\left(\sqrt{\frac{r}{R-r}}\right)\right] \,.\end{aligned} \tag{19.27}$$

This leads to a *very* different description of the infalling body's motion! Let's look at this function as $r \to 2GM/c^2 + x$: being very careful with our expansions, we find that as $x \to 0$,

$$t(x) \to \frac{2GM}{c^3}\ln\left[\frac{8GM(R - 2GM/c^2)}{Rc^2 x} + \mathcal{C}_1\right] + \mathcal{C}_2 \,. \tag{19.28}$$

The quantities $\mathcal{C}_{1,2}$ are constants whose precise values depend on the starting radius R, but are not important for us right now. In particular, note that the influence of the constant $\mathcal{C}_1$ becomes negligible as x gets small. Neglecting $\mathcal{C}_1$, we can easily rearrange this to find x as a function of t:

$$x \to \frac{8GM(R - 2GM/c^2)}{Rc^2}\exp\left[-(t - \mathcal{C}_2)c^3/(2GM)\right] \,. \tag{19.29}$$

The infalling body only asymptotically approaches $r = 2GM/c^2$ as $t \to \infty$.

To nail this home, let's plot the motion according to these two time parameterizations:

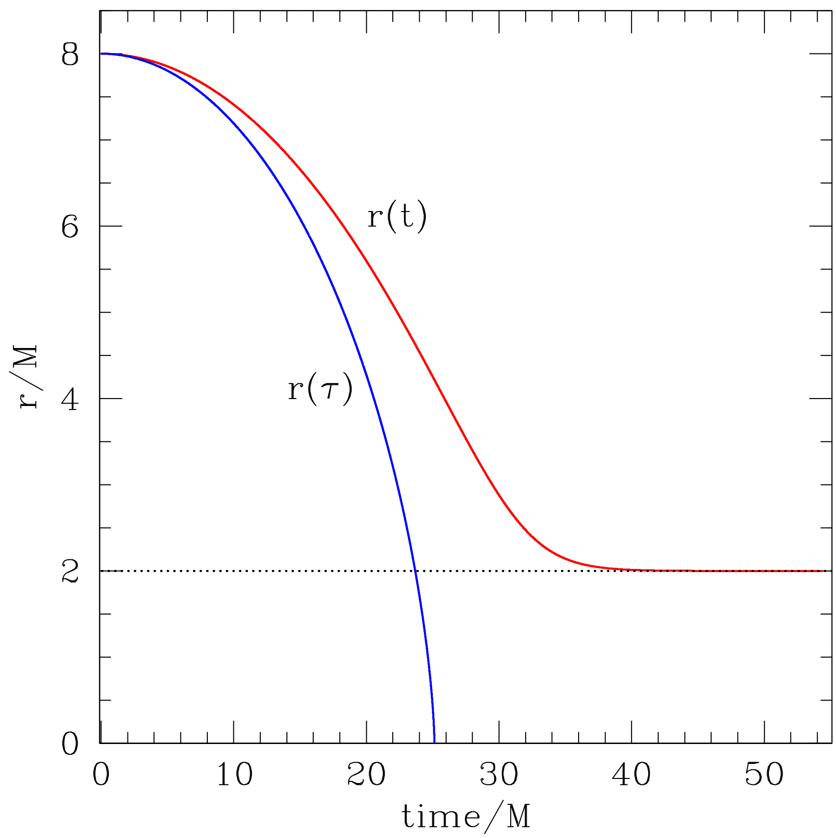

Figure 1: Infall trajectory from $R = 8GM/c^2$, parameterized by the infalling observer's time τ [using Eq. (19.24)] versus the trajectory parameterized by distant observer time t [using Eq. (19.27)]. Adapted from the course notes for 8.962; include a multiplicative factor of G/c^2 on the M on the vertical axis, and a factor of G/c^3 on the M on the horizontal axis.

We have two very different pictures: According to the observer's own proper time, they more or less plummet merrily along, reaching $r = 0$ in short order according to their own clocks. (Incidentally, gravity diverges at $r = 0$, so that's not a very happy place to wind up.) But according to the clocks of very distant observers, they never get anywhere *close* to $r = 0$. Indeed, they only asympotically approach $r = 2GM/c^2$, reaching it only as $t \to \infty$ according to those observers.

A favorite saying of Einstein's was *Raffiniert ist der Herr Gott, aber boschaft ist er nicht* — "Subtle is the Lord, but malicious he is not." This figure seems to reveal a side of Nature that is not only malicious but positively perverse. A driving principle throughout this course has been that the view of two different observers must be *consistent* — perhaps they differ in some details, but they agree on physical outcomes. Can we possibly reconcile these two vastly different viewpoints consistent?

The answer will be yes, and the reconciliation is subtle. Doing so will turn on thinking very carefully how the observer who is very far away is observing the infalling observer.

MIT OpenCourseWare
https://ocw.mit.edu

8.033 Introduction to Relativity and Spacetime Physics
Fall 2024

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.