Massachusetts Institute of Technology
Department of Physics
8.033 Fall 2024

Lecture 20
Exploring strong gravity

20.1 Overview

In this set of notes, we are going to explore some of the ways in which observational tests of the unique predictions of strong gravity can be formulated. These notes are a little on the long side, and will probably be delivered over the course of multiple lectures. They also rather dense, and involve calculations whose details require a bit of care. Every one of these calculations is, at heart, nothing more than an examination of a "geodesic" (the trajectory followed by a freely-falling body or the propagation of light) in the strong-field spacetime. The recipe for performing such a calculation is always the same — write down the Lagrangian for motion in the spacetime, apply the Euler-Lagrange equations. The core ideas underlying these calculations are hopefully clear to you; do not worry if the amount of information is a bit too much to follow. We summarize the key points that we hope you take away from this discussion at the end of these notes.

20.2 Weirdness of infall according to two different observers

Our discussion of motion in the Schwarzschild spacetime reached the point where we looked at an infalling observer: someone who is at rest at $r = R$, then falls. What we found is that this observer's motion as a function of time looks radically different depending on what "time" means. If we parameterize the observer's trajectory using proper time τ, i.e. the time that the observer measures on their own clocks, we find a trajectory that is a simple function relating their coordinate position r with their measured proper time τ. This parameterization shows that the infalling observer reaches $r = 0$ in finite proper time.

On the other hand, if we parameterize the observer's motion using coordinate time t, which describes time as measured on the clocks of observers who are very far away, we get a very different picture. With that parameterization, the infalling observer never crosses $r = 2GM/c^2$, let alone reaches $r = 0$. Instead, we see them asymptotically approaching $r = 2GM/c^2$ as $t \to \infty$. This behavior is shown in Fig. 1.

From the exact solution for $r(t)$ written down in the previous lecture, examine how things behave near $r = 2GM/c^2$. Putting $r = 2GM/c^2 + \delta r$, it is not too hard to show that

$$\delta r = \frac{8GM(R - 2GM/c^2)}{Rc^2} e^{-(t-\mathcal{C}_2)c^3/(2GM)} . \tag{20.1}$$

The constant $\mathcal{C}_2$ depends on the initial condition; its precise value is not important for us. What this expansion shows us is that at late times in the t parameterization, the infalling observer gets closer to $2GM/c^2$ by a factor of e for every time interval $2GM/c^3$. This is a very short time. For example, if M is the mass of the Sun, $2GM/c^3$ is roughly 10 microseconds.

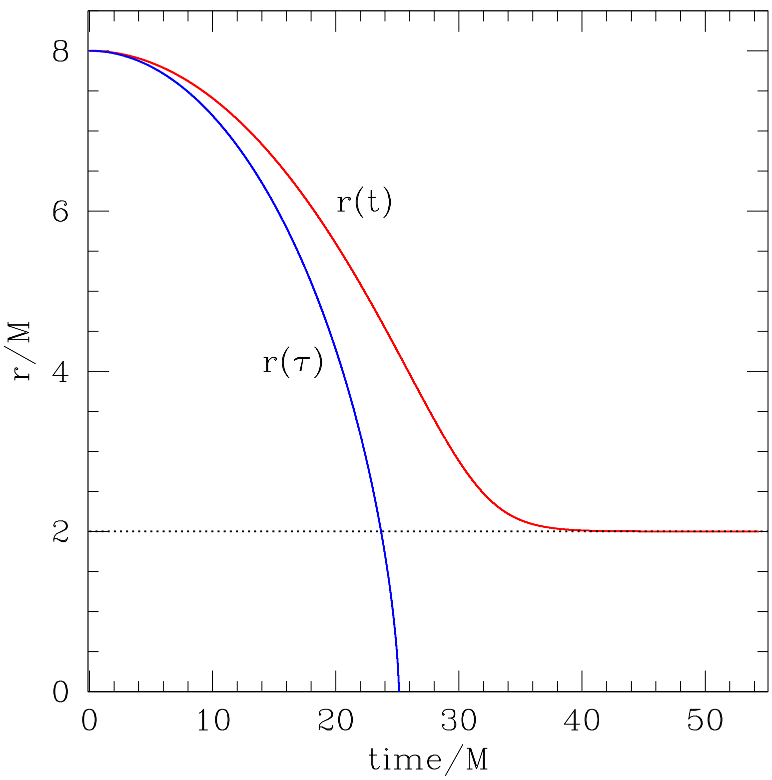

Figure 1: Infall trajectory for $R = 8GM/c^2$, parameterized by the infalling observer's proper time τ, and parameterized by distant observer time t. Figure taken from lecture notes for 8.962; include a multiplicative factor of G/c^2 to the M on the vertical axis, and a factor of G/c^3 to the M on the horizontal axis.

Throughout our study of relativity, we have encountered situations in which two observers measure different things. We've learned not to be too bothered by this, but have learned instead to try to find a way to make sure that measurements, though perhaps not in *agreement* (two observers measure different lengths and see events happen in a different order; one measures a pure magnetic field, another measures a combination of electric and magnetic fields) are nonetheless *consistent*. We have found that the observers agree on the nature of important events at the end of the analysis (a long pole moving very fast never collides with a door; an electron feels a force which causes it to accelerate).

But making consistent the two pictures of the infall-in-Schwarzschild scenario that we've painted seems like a tall order. How can we reconcile falling all the way to $r = 0$ in one parameterization (where, incidentally, gravitational tidal stresses get so strong that anyone or anything will be shredded) with "hovering" near $r = 2GM/c^2$ for all eternity in another?

We do this by thinking carefully about what the coordinate t means. An observer who is fixed at coordinate $r = R$ uses a clock that ticks uniformly in intervals of their proper time τ. Compared to intervals in t, that observer's clock behaves as

$$d\tau = dt\sqrt{1 - \frac{2GM}{Rc^2}}\ . \tag{20.2}$$

Notice that as $R \to \infty$, $d\tau \to dt$. In other words, the coordinate t is in fact proper time for observers who are at rest very far away from the mass M. This hopefully makes sense since, as we described in the previous lecture, the Schwarzschild metric looks just like the spacetime of special relativity (in spherical coordinates) for an observer who is very far away.

As we've emphasized a few times, the coordinate t describes the clocks of observers who are far away. How do these clocks communicate and "sync up" with the clocks of other observers? In special relativity, we did this using light — the invariant properties of light give us a tool which allows us to connect the clocks of different observers. General relativity inherits this: the different clocks of different observers are synchronized with one another by allowing their properties to be carried from observer to observer using light.

However — and this is the crucial point — *the propagation of light is strongly affected by gravity.* Clocks which tick very nicely in time t very far from the mass M do *not* tick so nicely when they are close to $r = 2GM/c^2$. In fact we, saw that light's coordinate velocity goes to zero as $r \to 2GM/c^2$. Light emitted at that radius carries no energy to observers who are infinitely far away.

This helps us to see why infall according to the t parameterization looks so weird compared to infall in the τ parameterization. In the τ parameterization, we are writing things in terms of a clock that makes sense exactly where the infalling observer is located. That parameterization in essence tells us exactly what the infalling observer is actually experiencing. The t parameterization, on the other hand, tells us how things "look" according to an observer who is watching things happen from very far away. **The distant observer makes their measurements using light** (or, as we'll discuss a little later, other forms of radiation that travel at the speed of light) **and so their measurements are affected by how gravity affects light.** In essence, the infalling observer never crosses $r = 2GM/c^2$ according to the t parameterization because the light that allows distant observers to see that happen never reaches them. It doesn't mean that they see this observer just "hovering" outside $r = 2GM/c^2$, however — they actually see *nothing at all.*

Imagine that as the infalling observer approaches this point, they carry a beacon which emits a signal — "My time is now τ, and all is well!" — directed to a distant observer. As the infalling observer approaches $r = 2GM/c^2$, the light takes longer and longer (according to very distant clocks) to get out. The message is also increasingly redshifted to longer and longer wavelengths. Keep in mind that at late times (as seen by distant observers), the infalling observer gets closer by a factor of e after every interval $\Delta t = c^3/2GM$. This time is a bit less than 10 microseconds if $M = 1$ solar mass. This means that for $M = 1$ solar mass, after an interval $\Delta t = 1$ second, the distant observer sees the position of the infalling observer change from $r = 2.01GM/c^2$ to $r = 2(1 + 0.01e^{-100{,}000})GM/c^2$. *Very* quickly we can no longer see the infalling observer, nor get any message from them.

Although strictly speaking, the distant observer claims that their infalling friend never crosses $r = 2GM/c^2$, when that friend gets close to this radius, what the distant observer claims does not matter. No communication with the infalling friend is ever possible. For all practical purposes, their friend has merged with the spacetime, and can no longer be distinguished as an independent entity. (Indeed, later measurements would show that the mass which appears in the spacetime line element is no longer M, but has become $M + m_{\rm friend}$. Your friend is part of spacetime now.)

20.3 The event horizon

A good way to summarize this discussion is that the time coordinate t is perfect for observers who are very far away. Indeed, t is their proper time, and is how they naturally describing the ticking of clocks. The coordinate t can be used for all $r > 2GM/c^2$, though it gets increasingly problematic as r gets close to $2GM/c^2$. It is bad exactly at $r = 2GM/c^2$.

What about for $r < 2GM/c^2$? To be blunt, this is tricky. Time t connects clocks from the very distant region to other places using light; since light doesn't propagate at all when $r = 2GM/c^2$, t simply ceases to be a useful measure of time that radius. This means that we need to be a lot more careful about how we set up and define "past" and "future" when we examine the region $r \leq 2GM/c^2$. This requires a bit more setup and analysis than is appropriate for 8.033, but we can borrow the punchline for our purposes: with a little effort, it can be shown that **no light ray** emitted at $r < 2GM/c^2$ can ever propagate to $r > 2GM/c^2$. In essence, the behavior that we saw at $r = 2GM/c^2$ — light rays emitted exactly at that spot remain forever bound to that spot — turns around. What we find is that all light rays inevitably evolve to smaller and smaller values of r. Even a light ray that we "think" is pointing outward ends up on a trajectory that eventually hits $r = 0$.

Since light can never "get out" from $r \leq 2GM/c^2$, this radius defines a boundary beyond which events cannot communicate with the rest of spacetime. We call this boundary the *event horizon* — events inside $r_H = 2GM/c^2$ are "over the horizon" and forever out of our reach. An object which has an event horizon is called a *black hole*.

The event horizon is one of the strangest predictions of physics. Before thinking about whether we can safely test for its existence, it's worth pausing to review a few issues that you might wonder about. Further discussion of these points can be found in a paper[1] which summarizes a presentation of these issues at a summer school for graduate students.

- The Schwarzschild spacetime describes an object that is spherically symmetric, and is non-rotating. Are conclusions about the nature of event horizons robust if they are made with such a "special" configuration?

  It turns out we do indeed need to go beyond this spacetime; the Kerr spacetime that we mentioned in Lecture 18 (which describes a rotating black hole) turns out to be just what we need. It describes an object which has rotation, and has an event horizon at radius $r_H = \tilde{M} + \sqrt{\tilde{M}^2 - a^2}$, where $\tilde{M} = GM/c^2$, $a = J/Mc$, and J is the magnitude of the object's spin angular momentum. Although the quantitative details change as we go from Schwarzschild to Kerr, the qualitative picture remains pretty much the same. Light rays at $r = r_H$ remain bound to that radial location (although they "twist" in axial coordinate, in essence being dragged along by the black hole's spin; you'll explore a related issues arising from this behavior on pset #9). Light rays emitted at $r < r_H$ can never reach $r > r_H$, but instead inevitably propagate to $r = 0$.

- Is Kerr the end of the discussion, or is there a whole "zoo" of spacetimes with event horizons that may describe black holes?

  There is one further change we can make beyond Kerr — black holes can have electric charge — but that is it. It is expected that in "real life," any black hole's charge will be vanishingly small, since they will tend to form near environments with a lot of free plasma. The combination of electromagnetic and gravitational forces means

[1] `https://arxiv.org/abs/hep-ph/0511217`

that black holes will tend to pull in charges opposite in sign to their own charge, neutralizing themselves and leaving a Kerr black hole behind. So a Kerr black hole is, for observational and experimental purposes, indeed the end of the discussion.

You might wonder: What happens if something disturbs this black hole? Won't it change its shape or its other properties in some way? Indeed, such a thing can happen (and we'll talk about some examples soon). However, a set of remarkable results prove what is now summarized as the *no-hair theorem*: A spacetime which contains an event horizon is either the Kerr solution (which does not change with time), or it is time evolving. *If it is time evolving, then it evolves into the Kerr black hole spacetime.* This time evolution forces the spacetime to "shake off" all deviations, until only the Kerr solution remains.

What this means is that if we have a black hole described by the Kerr spacetime and it is disturbed somehow (perhaps you threw your roommate into it to see if has an event horizon), it will "jiggle" a little bit, and then settle down to a new Kerr spacetime. Its mass and spin might change after the disturbance; the nature of the jiggling may tell us something about what happened to disturb it. We describe this as the black hole "no-hair" theorem because it tells us that black holes have no distinguishing structure (no "hair") beyond their mass and spin (and, in principle, their electric charge).

- The black hole spacetime exists for all time. Are the only black holes we might encounter in Nature ones that have existed forever? Or does Nature provide a way to make black holes from "normal" objects?

  It is not terribly difficult to show that, starting with normal-behaved matter, it can evolve into a spacetime with an event horizon. For a very special but unrealistic case, this can be done analytically: a spherical ball of dust, with no pressure to hold it up against gravity, collapses to form a black hole. This calculation was first done by the famous physicist Robert Oppenheimer with his student Hartland Snyder; their result appeared in publication on September 1 1939. (This date is important for other reasons; Oppenheimer's research moved into military applications very soon after this.) The Oppenheimer-Snyder collapse calculation is simple enough that I use it as a homework exercise in 8.962. For more realistic situations (including effects such as rotation and realistic pressure profiles), it requires numerical computation (though a UROP student is now exploring whether including rotation can be done in a simple way). The outcome of these studies is that we can indeed start with "normal" matter, and have it evolve into a black hole.

- What about the behavior as $r \to 0$? We argued earlier that, kind of like the Coulomb point charge, there must be an infinite amount of "stuff" crammed into zero volume there, at least classically. Can that behavior possibly be correct? Doesn't quantum physics have something to say about this?

  The nature of what happens at the very center of the black hole remains a mystery. Quantum effects must surely have a major impact on the nature of things as we approach $r \to 0$; the exact nature of those effects remains unknown since we haven't formulated an undisputed quantum theory of spacetime. This is a source of some concern. One thing that is indisputable is that everything which crosses the event horizon eventually reaches $r = 0$. Indeed, when we develop a better parameterization to describe "time" for a body that has crossed the horizon, we discover that $r = 0$ is not really the "center" of the spacetime. Instead, $r = 0$ is actually the *future* of the spacetime — at least, the future of all spacetime interior to $r = 2GM/c^2$. (Further discussion of this point can be found in the paper whose URL is listed in the footnote on a previous page.)

Whatever mysteries may occur as $r \to 0$, they are hidden from us by the black hole's event horizon, and cannot have any effect on measurements that we make out in the rest of the universe[2]. This gives us freedom to apply the laws of physics that we understand to the region of spacetime that is able to communicate with us. We content ourselves, for now, with the fact that physics we believe we understand describes everything that we can measure in principle[3].

20.4 What can we observe?

Putting these issues of principle to the side, the question becomes: what observations can we make of an object which tell us that the spacetime has the strong-field properties we expect if the object is a black hole? This is a very open-ended question (to first approximation, answering this question has constituted a substantial fraction of your lecturer's research career). In this discussion, we look at two aspects of motion in black hole spacetimes (focusing on the Schwarzschild case, which is relatively simple) that are hallmarks of the object being a black hole. Motion in these spacetimes is important because black holes are themselves totally dark. But, if things move near them, we may be able to measure the light that these objects emit — or some other kind of radiation associated with the motion.

20.4.1 The motion of material bodies

A material body moving near a Schwarzschild black hole is governed by the Lagrangian

$$L = \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = \frac{1}{2}\left[-\left(1 - \frac{2GM}{rc^2}\right)(c\dot{t})^2 + \frac{(\dot{r})^2}{(1 - 2GM/rc^2)} + r^2(\dot{\theta})^2 + r^2\sin^2\theta(\dot{\phi})^2\right] , \tag{20.3}$$

where $\dot{x}^\alpha = dx^\alpha/d\tau$. As you showed on problem set #8, because the spacetime is independent of t and independent of ϕ, we can find two constants of the motion right away. If we focus

[2]Should you be curious to read about this, the hypothesis that singularities are hidden behind event horizons is called the "cosmic censorship conjecture." Note that this is a conjecture, not a theorem. Indeed, counterexamples have been found, although they tend to describe very special circumstances that will not happen in Nature. From an experimental/observational standpoint, the idea that the $r = 0$ singularity is always "clothed" by an event horizon works well enough that many of us go along with it. We're certainly not 100% satisfied about the situation given that this conjecture remains just a conjecture.

[3]There is another lingering bit of concern, which is that when we apply the leading effects of quantum physics to black holes, we find that they lose mass, eventually evaporating away entirely. This is the phenomenon of *Hawking radiation.* What happens to the mysteries at $r = 0$ then? We do not have a full understanding of this, and it's quite bothersome. We can content ourselves with knowing that the timescale for evaporation is so long that for most black holes we encounter, evaporation is unlikely to be a problem. For instance, a black hole of 1 solar mass will take about 10^{67} years to evaporate, and this lifetime scales with mass cubed. We appear to have some time to figure this out.

on motion that is confined to the plane $\theta = \pi/2$, then these constants take the values

$$\frac{\partial L}{\partial \dot{t}} = -\left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t} \equiv -\hat{E}\ ; \tag{20.4}$$

$$\frac{\partial L}{\partial \dot{\phi}} = r^2 \dot{\phi} \equiv \hat{L}_z\ . \tag{20.5}$$

We interpret $\hat{E}$ as a conserved orbital energy (per unit mass), and we interpret $\hat{L}_z$ as a conserved orbital angular momentum (per unit mass); the z subscript[4] is because if we think of the normal to the orbital plane as the z axis, this is the angular momentum about that axis. (Since the orbit is confined to this plane, this is also the total angular momentum.)

We also know that $\vec{u} \cdot \vec{u} = g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = -c^2$, so (again using $\theta = \pi/2$)

$$-c^2 = -\left(1 - \frac{2GM}{rc^2}\right)(c\dot{t})^2 + \frac{(\dot{r})^2}{(1 - 2GM/rc^2)} + r^2(\dot{\phi})^2\ . \tag{20.6}$$

Using Eqs. (20.4) and (20.5), we can replace $\dot{t}$ and $\dot{\phi}$ in this expression, and rewrite it as an equation governing the radial motion of the body orbiting in this spacetime:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{\hat{E}^2}{c^2} - V_{\rm eff}(r)\ , \tag{20.7}$$

where

$$V_{\rm eff}(r) = \left(1 - \frac{2GM}{rc^2}\right)\left(c^2 + \frac{\hat{L}_z^2}{r^2}\right)\ . \tag{20.8}$$

This function is often called the "effective potential" for orbits in the Schwarzschild spacetime, since it plays the same role in determining the motion of bodies as a similar potential that is often used to describe Newtonian orbits, and from which we derive Kepler's laws.

Equations (20.7) and (20.8) are going to be our main tools for a little while, so it is worthwhile focusing on what they tell us. Figure 2 shows an example of what this function looks like, plotted for a particular choice of $\hat{L}_z$ (in the case, $\hat{L}_z = 3\sqrt{3/2}GM/c$).

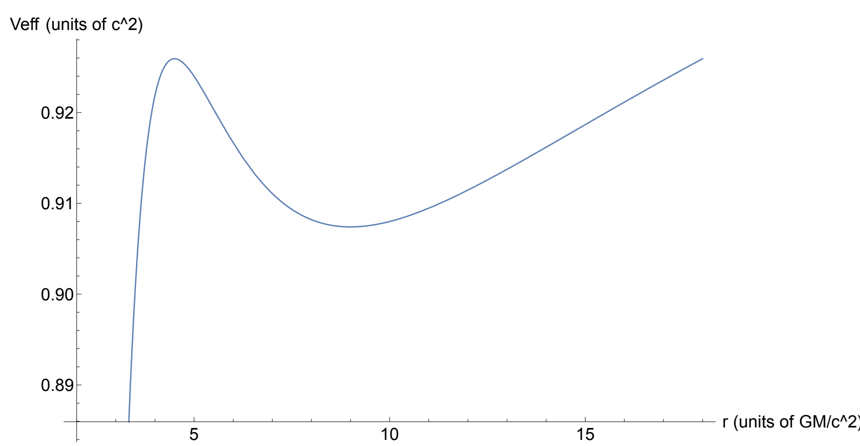

Figure 2: The function $V_{\rm eff}(r)$ plotted for $\hat{L}_z = 3\sqrt{3/2}GM/c$.

[4]It is also useful to "decorate" the symbol for angular momentum a bit so that it isn't too easy to confuse it with the Lagrangian.

Turn now to Eq. (20.7). This equation tells us that the radial component of the orbiting body's 4-velocity is determined by subtracting $V_{\rm eff}$ from a quantity made from the orbit's conserved energy, $\hat{E}^2/c^2$. There is a tremendous amount of information in this equation. We are going to use it to figure out how the body's motion depends on its energy and its angular momentum.

Consider for example the situation illustrated in Fig. 3.

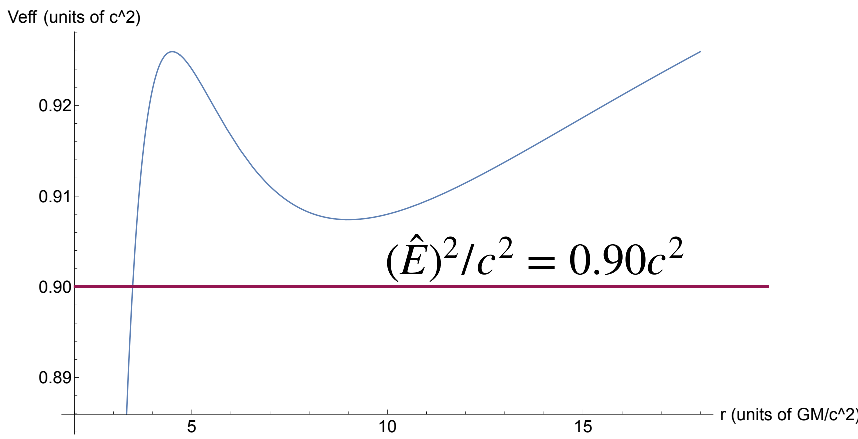

Figure 3: The same $V_{\rm eff}(r)$, showing this potential versus $\hat{E}^2/c^2 = 0.90c^2$.

For almost the entire span of radius we have included here, $\hat{E}^2/c^2$ is less than $V_{\rm eff}(r)$. This means that over this range $(dr/d\tau)^2$ is negative, and there is no real solution describing a body moving over these radii. At least for this value of angular momentum, $\hat{E} = \sqrt{0.9}c^2$ does not yield any allowed orbital motion.

Consider next the situation shown in Fig. 4. This is the same potential, but the energy is now $\hat{E}^2/c^2 = 0.924c^2$. Notice that this time $\hat{E}^2/c^2 \geq V_{\rm eff}$ over a range of radii.

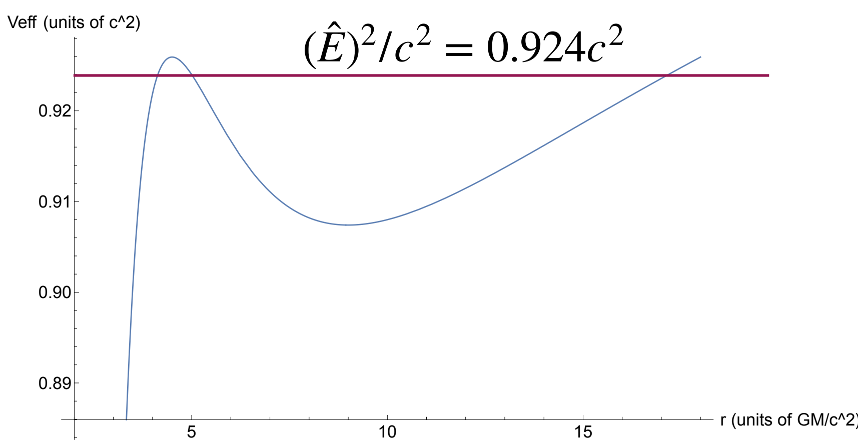

Figure 4: The same $V_{\rm eff}(r)$, showing this potential versus $\hat{E}^2/c^2 = 0.924c^2$.

For the situation shown in Fig. 4, $(dr/d\tau) \geq 0$ over a range of radii. This situation describes an *eccentric orbit*: an orbit that oscillates from $r_{\rm min} = 5GM/c^2$ to $r_{\rm max} \simeq 17.2GM/c^2$ and back over the course of its orbit. The turning points are defined by the condition $(dr/d\tau) = 0$, which means that they are found by finding the values of r at which $\hat{E}^2/c^2 = V_{\rm eff}(r)$. Bear in mind that while it is sloshing back and forth in radius, its angle ϕ is continually increasing: from the relationship between the conserved angular momentum $\hat{L}_z$ and $d\phi/d\tau$, we find

$$\frac{d\phi}{d\tau} = \frac{\hat{L}_z}{r^2} \; . \tag{20.9}$$

In the Newtonian limit, this orbit would look like a closed ellipse if we looked at its angular and radial motion together. In general relativity, the orbit does not quite close, and we get a more interesting pattern. Defining $x = r(\tau)\cos[\phi(\tau)]$, $y = r(\tau)\sin[\phi(\tau)]$, Fig. 5 shows one complete radial period for the parameters used in Fig. 4:

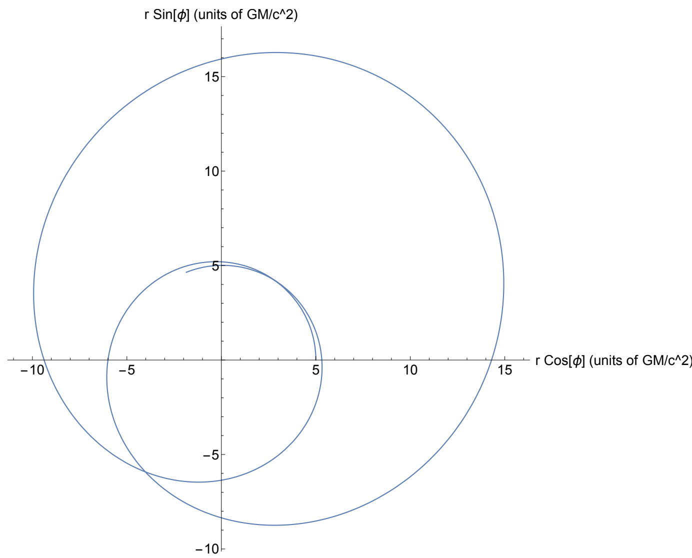

Figure 5: Motion in r and ϕ for an orbit with $\hat{L}_z = 3\sqrt{3/2}GM/c$, $\hat{E} = 0.924c^2$.

This is a "strong-field" orbit; it looks *very* different from a Newtonian orbit. In particular, the orbit moves through a *lot* more than 2π radians of ϕ in the time in takes to move from $r_{\rm min}$ to $r_{\rm max}$ and back. (This example actually completes 2.31 complete "whirls" in ϕ during a single cycle of radial motion.) When the orbital radius is at all times large compared to GM/c^2, the "extra" ϕ per orbit is much smaller. In fact, it is not too hard to show that this motion precisely reproduces the anomalous precession of Mercury's orbit that so excited Albert Einstein in 1915.

A particularly special value of the energy is illustrated in Fig. 6. This value is chosen so that $\hat{E}^2/c^2 = V_{\rm eff}(r)$ at exactly one point. This case defines a *circular orbit*. To find this orbit, we require that $\hat{E} = c\sqrt{V_{\rm eff}(r)}$ (so that $dr/d\tau = 0$). We also require that the orbit "live" at the minimum of the effective potential: $\partial V_{\rm eff}/\partial r = 0$. As you show on problem set #9, this yields a set of analytic solutions that describe the energy and angular momentum per unit mass for a body in a circular orbit:

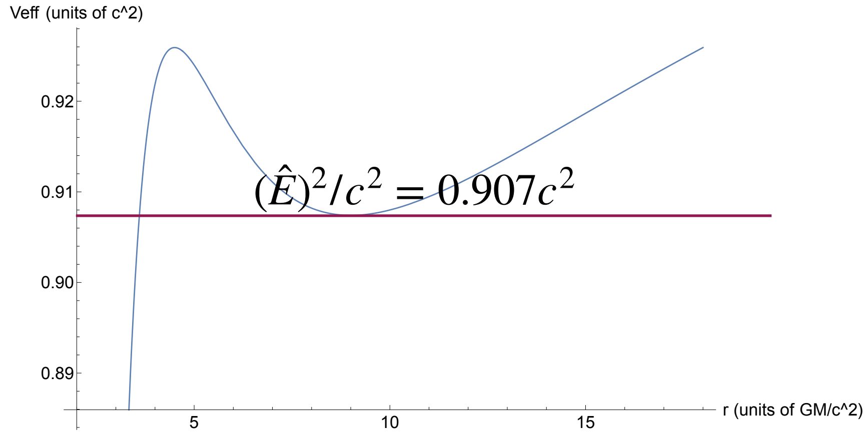

Figure 6: The same $V_{\rm eff}(r)$, showing this potential versus $\hat{E}^2/c^2 = 0.907c^2$.

$$\hat{L}_z = \pm\sqrt{\frac{GMr}{1 - 3GM/rc^2}}\ , \quad \hat{E} = \frac{1 - 2GM/rc^2}{\sqrt{1 - 3GM/rc^2}}\ . \tag{20.10}$$

(The $\pm$ on the angular momentum is because we take a square root at one point, and both solutions are valid. The signs describe orbits going in opposite ϕ directions at radius r.)

One last point before moving on: if we examine a sequence of potentials, we find something interesting and a little odd. For almost all values of $\hat{L}_z$, the potential qualitatively has the shape we saw in the previous figures — a peak at small radius, with a minimum at some finite r, the asymptoting to $V_{\rm eff} \to c^2$ as $r \to \infty$. However, if we make $\hat{L}_z$ small enough, we see that there is a change. Figure 7 shows what happens as we reduce $\hat{L}_z$ from $3.5GM/c$ to $3.45GM/c$:

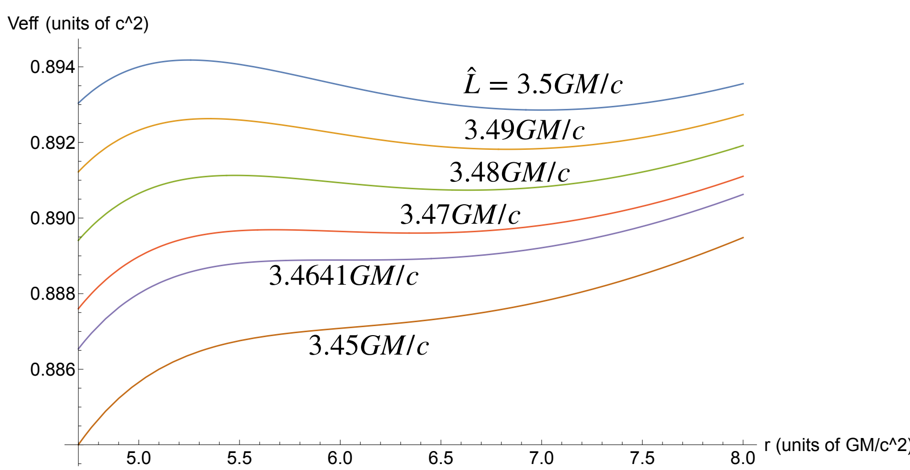

Figure 7: A zoom onto the region of the minimum for a sequence of $V_{\rm eff}(r)$, looking at different values of $\hat{L}_z$.

As $\hat{L}_z$ gets smaller, the minimum moves to smaller values of r. At the same time, the vicinity of the minimum gets progressively flatter, and is less well defined as a minimum. In fact, at a certain point the function flattens out so much that *the minimum goes away altogether.* When this happens, **there are no more stable circular orbits**.

To find when the minimum goes away, we look for the point along the sequence at which the first and second derivatives both vanishes: $\partial V_{\rm eff}/\partial r = 0$ and $\partial^2 V_{\rm eff}/\partial r^2 = 0$ at the same value of r. A quick analysis shows us that this condition is met when

$$\hat{L}_z = \sqrt{12}GM/c \simeq 3.4641GM/c\ . \tag{20.11}$$

(The purple curve in Fig. 7 was computed with exactly this special value of $\hat{L}_z$.) Using the formula that relates a circular orbit's energy to its radius tells us that this happens when $r = 6GM/c^2$. Here is a prediction about strong-field orbits that is *starkly* different from what we encountered with Newtonian gravity: **No stable circular orbits exist at all for radii $r \le 6GM/c^2$.**

20.4.2 The motion of light

On problem set #9, you looked at how light is bent by gravity in the weak gravity limit. What is the strong-field analog of this?

To answer this question, let's think about the geometry of a light ray that comes in from very far away. Let's imagine that the black hole is at the origin of coordinates, and the light ray comes in parallel to the x axis. Far away, it is displaced from the axis by a distance b. We will call this distance the *impact parameter* of the incoming light ray. From basic mechanics, we can say that the light ray has an angular momentum that is related to the x component:

$$|\mathbf{L}| = |\mathbf{r}\times\mathbf{p}| = bp^x = bE/c \equiv L_z\ . \tag{20.12}$$

All of the quantities in this equation are evaluated very far away, where the spacetime is the same as that of special relativity. The geometry of this situation is illustrated in Fig. 8.

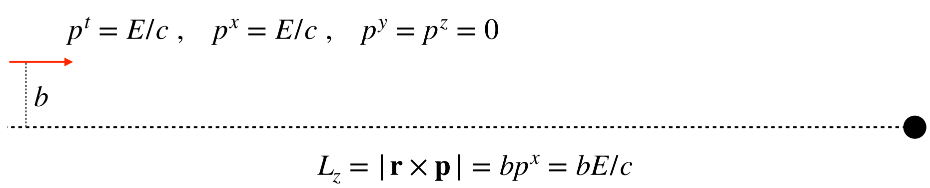

Figure 8: The geometry of a photon that is "launched" from far away toward the black hole.

To figure out how the photon evolves as it propagates in to the strong gravity region, consider the Lagrangian for light:

$$L_{\rm light} = \frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \frac{1}{2}\left[-\left(1-\frac{2GM}{rc^2}\right)\left(c\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2\right]\ . \tag{20.13}$$

(We've simplified this a bit by putting $\theta = \pi/2$, $d\theta/d\lambda = 0$, and we are using $\dot{x}^\alpha = dx^\alpha/d\lambda$.) Because $\partial L_{\rm light}/\partial t = 0$ and $\partial L_{\rm light}/\partial\phi = 0$, we identify both $\partial L_{\rm light}/\partial\dot{t}$ and $\partial L_{\rm light}/\partial\dot{\phi}$ as constants of the motion:

$$\frac{\partial L_{\rm light}}{\partial \dot t} = -\left(1 - \frac{2GM}{rc^2}\right)c^2\frac{dt}{d\lambda} = c\,g_{tt}p^t = c\,p_t \equiv -E\;; \tag{20.14}$$

$$\frac{\partial L_{\rm light}}{\partial \dot\phi} = r^2\frac{d\phi}{d\lambda} \equiv L_z = bE/c\;. \tag{20.15}$$

In setting these equalities, we've used the fact that it is very easy to compute p_t and L_z very far from the black hole. But, because they are constants along the light ray, once we've computed them, we can use these values through the entire calculation.

Using $\vec p\cdot\vec p = 0$ in combination with these relations between E, $dt/d\lambda$, L_z, and $d\phi/d\lambda$ we find an equation for $dr/d\lambda$ that is similar in form to the equation we found for the motion of a material body:

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{E^2}{c^2} - \frac{L_z^2}{r^2}\left(1 - \frac{2GM}{rc^2}\right)\;. \tag{20.16}$$

This is a useful form because it parameterizes the motion in terms of the constants of motion L_z and E. However, we can do better, since we know that $L_z = bE/c$. Let's use this to eliminate E from the equation:

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{L_z^2}{b^2} - \frac{L_z^2}{r^2}\left(1 - \frac{2GM}{rc^2}\right)\;. \tag{20.17}$$

And, since L_z is itself a constant of the motion, we can eliminate it from the right-hand side:

$$\begin{aligned}\frac{1}{L_z^2}\left(\frac{dr}{d\lambda}\right)^2 &= \frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{rc^2}\right)\\ &= \frac{1}{b^2} - V_{\rm light}(r)\;. \end{aligned}\tag{20.18}$$

This equation now tells us that light propagates whenever $1/b^2 > V_{\rm light}$, where $V_{\rm light}$ plays a role in our analysis just like the effective potential that governs the motion of material bodies. However, $V_{\rm light}$ is much simpler — it doesn't depend on *any* free parameters.

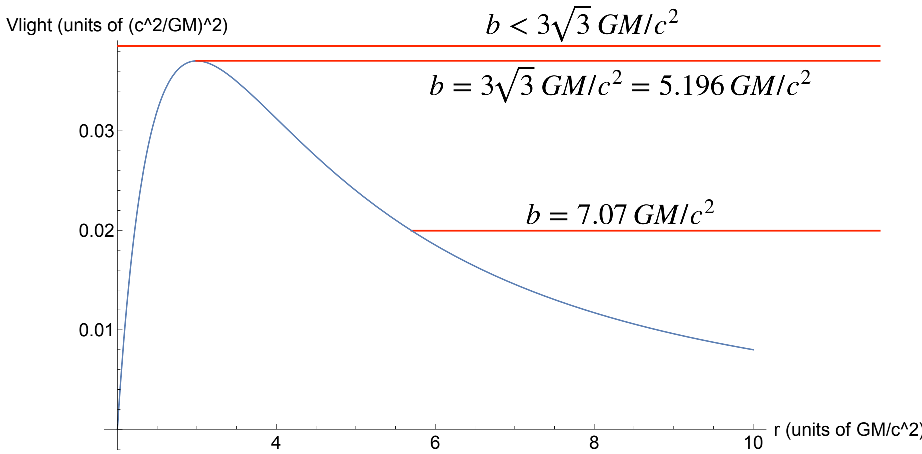

Figure 9: The potential $V_{\rm light}(r)$ that governs the motion of light in the Schwarzschild spacetime, plus a few lines illustrating $1/b^2$ for several values of b.

Figure 9 plots this potential as a function of r. Its maximum occurs at $r = 3GM/c^2$; its value at the maximum point is given by

$$V_{\rm light}(3GM/c^2) = \left(\frac{c^2}{GM}\right)^2 \left[\frac{1}{9}\left(1 - \frac{2}{3}\right)\right] = \frac{c^4}{27G^2M^2} = \left(\frac{c^2}{3\sqrt{3}GM}\right)^2 . \tag{20.19}$$

Figure 9 also includes several lines illustrating $1/b^2$ for several interesting values of b. Light propagates in radius as long as $(dr/d\tau)^2 > 0$, which occurs when $1/b^2 > V_{\rm light}$. With that in mind, let's examine how light behaves in this spacetime for several values of b:

- If $b > 3\sqrt{3}GM/c^2$, light propagates in from infinity just fine until it reaches the radius at which $1/b^2 = V_{\rm light}$. At this point, the light reverses radial direction, and heads back out to larger radius. When we look at the motion in the equatorial plane, we see that it comes in, has its trajectory bent as it comes closest to the black hole, then heads back out to large radius. An example of this is shown in the top-left of Fig. 10.

- If $b < 3\sqrt{3}GM/c^2$, light propagates from infinity all the way to $r = 0$. In this case, light passes into the event horizon and disappears from us forever. This is illustrated in the top-right of Fig. 10.

- If $b = 3\sqrt{3}GM/c^2$, light comes in until it reaches $r = 3GM/c^2$. At this point, $dr/d\tau = 0$, so it sits in this *light orbit* forever. This is an unstable orbit, so the slightest deviation from $b = 3\sqrt{3}GM/c^2$ means that the light will either eventually zoom away or else fall in. The bottom of Fig. 10 shows what happens when b is too large by $4 \times 10^{-8}GM/c^2$. In this case, the light completed about 3.7 orbits before zooming away.

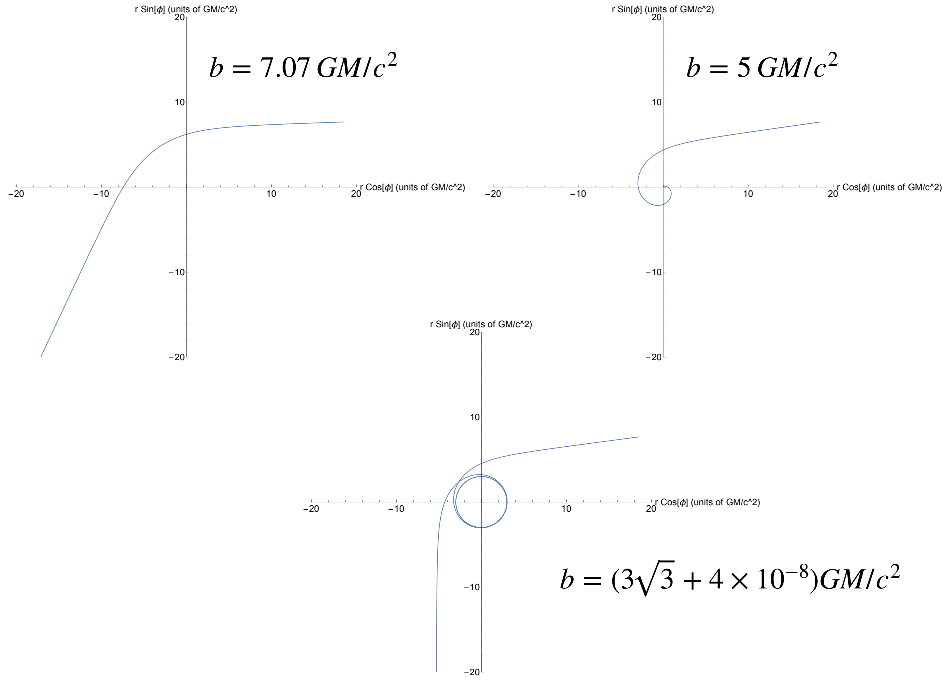

Figure 10: Examples of the (r, ϕ) motion for light propagating in the Schwarzschild spacetime.

This light orbit, or *light ring* is a distinctly non-Newtonian feature. But, it turns out to be something that makes a very strong observational prediction. Imagine that a black hole in Nature is immersed in some kind of hot gas or plasma. That material will emit electromagnetic radiation across a wide range of wavelengths. If some of that radiation passes close to the black hole hole, we expect that its trajectory will be bent very strongly around the black hole. Some of it will even get "trapped" into the light orbit at radius $r = 3GM/c^2$, at least for several orbits. When that radiation escapes from the light orbit, it can propagate toward us. The expectation is that we could see a ring of radius $3\sqrt{3}GM/c^2$ on the sky. This has long been regarded as a "smoking gun" of the strong-field nature of a black hole spacetime.

It turns out other kinds of radiation get trapped at that radius too, and leave an imprint that likewise is a smoking gun of the nature of the spacetime. We will discuss that in the next lecture.

20.5 Summary: Features of strong-gravity motion

This set of lecture notes is extremely dense. I hope that everyone who takes 8.033 is now capable of going through and understanding everything that is presented here; however, there is so much material (and this comes at such a stressful point in the semester) that I have little illusion that everyone will actually do that. So let me summarize what I regard as the most important points to note:

- Infall parameterized by τ looks completely different from infall parameterized by t. This is because t describes clocks that are very far away; they connect to clocks at small r using the trajectories of light. As one approaches $r = 2GM/c^2$, the propagation of light is hugely affected by gravity. The observer who is far away never sees an infalling observer cross $r = 2GM/c^2$ because the light they would use to make this measurement never reaches them.

- The radius $r = 2GM/c^2$ marks an *event horizon* in the spacetime. Events at $r < 2GM/c^2$ cannot communicate with events "outside the horizon." Once any trajectory crosses this radius, it is doomed to eventually reach $r = 0$, where it will find a quick (as measured by the trajectory's own clock) and violent end.

- The Lagrangian for the orbits of material bodies in this spacetime can be studied in the usual way with the Euler-Lagrange equations. Given an orbit's angular momentum per unit mass $\hat{L}_z$ and its energy per unit mass $\hat{E}$, one can figure out the kind of motion one is likely to get. We examined one example of an "eccentric" orbit (which generalizes the "elliptical" orbits of Newtonian gravity), and characterized circular orbits (which you will examine on pset #9).

- Stable circular orbits do not exist for $r < 6GM/c^2$, starkly non-Newtonian behavior.

- Light can be bent so strongly by gravity that it forms an orbit at $r = 3GM/c^2$. Observers far away may be able to see a ring of light with radius $b = 3\sqrt{3}GM/c^2$. This value of b is set by the impact parameter that puts light exactly on the light ring.