MIT Course 8.033, Fall 2005, Schwarzschild metric & black holes

Max Tegmark

Last revised November 29 2005

Key formula summary

- Schwartzschild metric in Gullstrand-Painlevé (GP) coordinates:

$$d\tau^2 = dt_{\rm ff}^2 - (dr + \beta_r dt_{\rm ff})^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

- Escape velocity $-dr/dt_{\rm ff}$:

$$\beta_r \equiv \left(\frac{2M}{r}\right)^{1/2}$$

- Schwartzschild metric in standard coordinates:

$$\begin{aligned} d\tau^2 &= \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right), \\ &= \gamma_r^{-2}dt^2 - \gamma_r^2 dr^2 - r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right), \end{aligned}$$

$$\gamma_r \equiv \frac{1}{\sqrt{1-\beta_r^2}}.$$

- Schwarzschild radius: $r = 2M$, corresponding to $\beta_r = 1$ and $\gamma_r = \infty$.

- Gravitational redshift for a clock at fixed r:

$$d\tau = \frac{dt_{\rm ff}}{\gamma_r} = \frac{dt}{\gamma_r}$$

- Effective potential:

$$\tilde{V}(\tilde{L}, r)^2 = \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)$$

- Equations of motion:

$$\begin{aligned} \left(\frac{dr}{d\tau}\right)^2 &= \tilde{E}^2 - \tilde{V}(\tilde{L}, r)^2, \\ \frac{d\varphi}{d\tau} &= \frac{\tilde{L}}{r^2}, \\ \frac{dt}{d\tau} &= \gamma_r^2\tilde{E} \end{aligned}$$

Radial motion:

$$\frac{dr}{d\tau} = \pm\sqrt{\beta_r^2 + \tilde{E}^2 - 1}$$

Circular orbits:

$$\frac{d\tilde{V}}{dr} = 0$$

- Photon orbits:

$$\begin{aligned} \left(\frac{dr}{dt}\right)^2 &= \gamma_r^{-4}\left[1 - \left(\frac{b}{\gamma_r r}\right)^2\right] \\ r\frac{d\varphi}{dt} &= \pm\frac{b}{r\gamma_r^2}. \end{aligned} \tag{1}$$

Small deflection of light:

$$\Delta\varphi \approx \frac{4M}{b}$$

when $b \gg M$.

The Schwarzschild metric

- Schwartzschild metric in Gullstrand-Painlevé (GP) coordinates:

$$d\tau^2 = dt_{\rm ff}^2 - (dr + \beta_r dt_{\rm ff})^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \qquad (2)$$

  where the escape velocity

$$\beta_r \equiv \left(\frac{2M}{r}\right)^{1/2}$$

  (measured by a "shell" observer at fixed r). Note that this happens to equal the Newtonian formula for escape velocity.

- Schwartzschild metric in standard coordinates:

$$\begin{aligned} d\tau^2 &= \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \\ &= \gamma_r^{-2}dt^2 - \gamma_r^2 dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \end{aligned} \qquad (3)$$

  where

$$\gamma_r \equiv \frac{1}{\sqrt{1-\beta_r^2}}.$$

- The GP form has the advantage of being valid for all $r > 0$, whereas the standard form is valid only for $r > 2M$, *i.e.*, outside the Schwarzschild radius $r_s \equiv 2M$.

- For both forms of the metric, θ and φ are the usual polar coordinates and r is defined as the proper circumference of a circle around the origin divided by 2π.

- The only difference between the two forms of the Schwarzschild metric is the choice of time coordinate. Both t and $t_{\rm ff}$ are "far-away time" in the sense that they tick at the rate you'd actually see time flow on a far-away clock at $r \approx \infty$ if you had a good enough telescope to see it. They differ by an additive factor, corresponding to how they are synchronized against this ficticious far-away reference clock. In both cases, you need to add to the time you see on the far-away clock an r-dependent offset $\Delta t(r)$ taking into account the time required for the light to reach you. t and $t_{\rm ff}$ correspond to two different choices of $\Delta t(r)$:

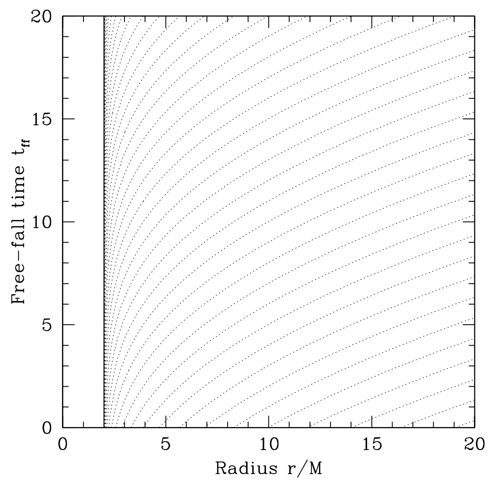
  - For t, sometimes called *bookkeeper's time*, $\Delta t(r)$ is half the round-trip light travel time between r and the reference clock (both ways take the same amount of t-time, since the metric above has no $dtdr$ cross-term).
  - For $t_{\rm ff}$, known as the *free fall time*, the synchronization is instead performed by letting a portable clock free-fall radially from rest at the far-away reference clock (where it has been synchronized) to r.

Relation between t and $t_{\rm ff}$

- These two forms of the Schwarzschild metric given by equation (2) and equation (3) are equivalent, related (for $r > 2M$ where the t-coordinate is well-defined) by the coordinate transformation

$$\begin{aligned} t_{\rm ff} &= t + \left(\frac{1}{\beta_r} - \tanh^{-1}\beta_r\right) 2M. \\ &= t + \left[\frac{1}{\beta_r} + \frac{1}{2}\ln\left(\frac{1-\beta_r}{1+\beta_r}\right)\right] 2M \end{aligned}$$

  You will prove this on problem set 9. The plot shows contours of constant t — note that t is undefined and hence useless for $r < 2M$.



- **Proof summary:** Differentiating the last equation gives

$$dt_{\rm ff} = dt + \beta_r\gamma_r^2 dr,$$

  which implies that

$$dr + \beta_r dt_{\rm ff} = \beta_r dt + \gamma_r^2 dr.$$

  Substituting this into equation (2) and simplifying gives equation (3).

- Let's prove that the GP coordinates have the interpretation we claimed, *i.e.*, that clocks free-falling radially past r from rest at infinity have velocity $-\beta_r$ and that their proper time interval $d\tau$ equals $dt_{\rm ff}$.
  Proof: A radial ($d\Omega = 0$) trajectory $r(t_{\rm ff})$ with $\frac{dr}{dt_{\rm ff}} = -\beta_r$ is one of maximal aging and hence a geodesic and a solution to the law of motion, since equation (2) gives

$$\Delta\tau = \int d\tau = \int \sqrt{1 - \left(\frac{dr}{dt_{\rm ff}} + \beta_r\right)^2} dt_{\rm ff} \le \int dt_{\rm ff} = \Delta t_{\rm ff},$$

  with equality only when the squared term vanishes, *i.e.*, when $\frac{dr}{dt_{\rm ff}} = -\beta_r$. This trajectory corresponds to clocks at rest at infinity since their infall speed $-\beta_r \to 0$ as $r \to \infty$.

Interpretating the Schwarzschild metric: the River Model

A natural interpretation of equation (2) (Hamilton 2004) is that space is flowing radially inward with a velocity β_r, and that particles can travel through this moving space according to the laws of special relativity, no faster than the speed of light (just like fish swimming through a flowing river with a maximum swim speed relative to the water). This is analogous to the FRW coordinates, where the "river" of space was expanding rather than flowing.

- **Event horizon:** At the Schwarzschild radius $r = 2M$, the river velocity equals the speed of light ($\beta_r = 1$), so for $r \leq 2M$, light can only move inward, not outward. Solving $d\tau = 0$ for radial light rays ($d\Omega = 0$) gives speeds

$$\frac{dr}{dt_{\rm ff}} = -\beta_r \pm 1,$$

  so even "outgoing" light rays (corresponding to the plus sign above) are in fact moving inward.

- **Black hole:** An object that lies within its own event horizon. (Idea due to Mitchell & Laplace, math due to Schwarzschild, Kerr, Reissner & Nordström, name due to Wheeler.) For rotating (Kerr) and electrically charged (Reissner-Norbström) black holes, the metric is more complicated.

- **Tidal forces:** Since the river is picking up speed as it flows inward, infalling objects are stretched in the radial direction.
  Since the river is converging (flowing radially towards the origing), infalling objects are compressed in the transverse (θ and ϕ) directions.

- **Infalling clocks:** Our ficticious infalling synchronization clocks are at rest relative to the space around them and hence stay at rest relative to the space around them and experience no special-relativistic time dilation ($d\tau = dt_{\rm ff}$). They are analogous to comoving galaxies in the FRW metric, since they stay at rest relative to space.

- **Singularity:** Since everything inside the event horizon is unavoidably pulled inward with ever greated speed, it will end up at $r = 0$ within a finite time (a time of order M), creating an infinite density at this point. This is called a singularity. We don't know what actually happens there and some physicists think that if we one day discover a theory of quantum gravity, it may replace the singularity by something finite and calculable.

- **Gravitational redshift and shell time:** Consider clocks attached to fixed concentric shells at different r. The time measured by such a clock at fixed position (r, θ, φ) is

$$dt_{\rm shell} = d\tau = \gamma_r^{-1} dt_{\rm ff} = \gamma_r^{-1} dt.$$

This means that this clock will run slow by a factor γ_r compared to a far-away clock. We can interpret this result $d\tau = dt_{\text{ff}}/\gamma_r$ as a Doppler shift, caused by our moving with speed β_r againt the inflowing river of space.

- **Why the t-coordinate breaks down:** The problem with the t-coordinate is that its definition requires sending a signal to the far-away clock, which is impossible from $r < 2M$. When you're "at rest" at the constant radius $r = 2M$, you see the far-away clock infinitely blueshifted because you are moving with the speed of light against the inflowing river of space, so you can interpret the problem as a Doppler shift caused by extreme, unnatural and (for massive observers) impossible motion.

- **Flatness:** In the GP coordinates, the Schwarzschild metric is spatially flat, since at constant free-fall time t_{ff} (take $dt_{\text{ff}} = 0$), the metric reduces to

$$d\tau^2 = -dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

i.e., simply the metric of 3D Euclidean space in spherical coordinates. This means that r is simply the radial distance to the origin measured along a spacelike curve of constant t_{ff}, so that the radius of a circle is its circumference over 2π — in contrast, the radial distance measured along a spacelike curve of constant t is longer and blows up at $r = 2M$.

- **Shell radius:** At fixed free-fall time t_{ff}, things are simple: the radial distance between two shells is simply dr.
At fixed bookkeeper time t, things are more complicated: the radial distance between two shells is dr/γ_r.
It defines what the book calls shell radius $dr_{\text{shell}} = dr/\gamma_r$, which is the distance you'd actually measure with a measuring between two shells. Integrating radially outward from the Schwarzschild radius at fixed bookeeper time t gives

$$\begin{aligned}\Delta r_{\text{shell}} &= \int dr_{\text{shell}} = \int_{2M}^{r} \gamma_{r'} dr' = \\ &= r\gamma_r^{-1} + M\ln\left[\frac{r}{M}\left(1+\gamma_r^{-1}\right) - 1\right]\end{aligned}$$

This is well-behaved and finite for any $r \geq 2M$ and gives $\Delta r_{\text{shell}} \approx r + M\ln\frac{r}{M}$ for $r \gg 2M$.

Radial light rays

- In any metric, light rays have $d\tau = 0$. For a photon moving radially in the GP metric, we have

$$\frac{dr}{dt} = -\beta_r \pm 1.$$

Integrating

$$dt = -\frac{dr}{\beta_r \mp 1}$$

gives the spacetime trajectories of outgoing ($-$) and infalling ($+$) radial light rays.

- **Ingoing photon:**

$$t_{\rm ff} = t_{\rm ff}^* - 2M\left[\beta_r^{-2} - 2\beta_r^{-1} + 2\ln(1+\beta^{-1})\right],$$

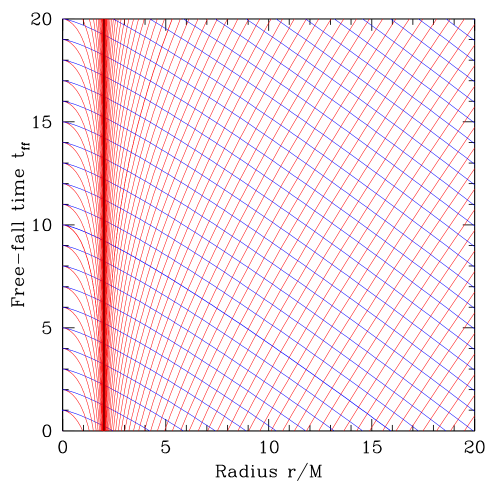
where $t_{\rm ff}^*$ is the time the photon hits the $r = 0$ singularity.

- **"Outgoing" photon:**

$$t_{\rm ff} = t_{\rm ff}^* + 2M\left[\beta_r^{-2} + 2\beta_r^{-1} + 2\ln(|\beta^{-1} - 1|)\right],$$

where $t_{\rm ff}^*$ is the time the photon hits the $r = 0$ singularity (for the $r < 2M$ case) or $t_{\rm ff}^* + 16M$ is the time the photon passes $r = 8M$ (for the $r > 2M$ case where the photon really is outgoing).

- The plot shows both ingoing photons (blue curves) and "outgoing" photons (red curves). Note that the ingoing ones cross the event horizon, but outgoing ones don't.



Orbits

- For a particle moving in the Schwarzschild metric, the energy E and and angular momentum L are conserved. It's convenient to divide these two by the rest mass of the particle and work with the energy per unit rest enery $\tilde{E} \equiv E/m$ (dimensionless, since $c = 1$) and the angular momentum per unit rest mass, $\tilde{L} \equiv L/m$ (units of length).

- In terms of these two constants, the equations of motion become

$$\begin{aligned} \left(\frac{dr}{d\tau}\right)^2 &= \tilde{E}^2 - \tilde{V}(\tilde{L}, r)^2, \\ \frac{d\varphi}{d\tau} &= \frac{\tilde{L}}{r^2}, \end{aligned}$$

where the *effective potential* per unit rest mass is

$$\tilde{V}(\tilde{L}, r)^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)$$

and the proper time τ is related to the t-coordinate in the Schwarzschild metric (far-away time) by

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - 2M/r} = \gamma_r^2 \tilde{E}.$$

- $\tilde{E} \geq 1$ is a neccessary condition for being able to escape to $r = \infty$ (where $\tilde{V} = 0$).

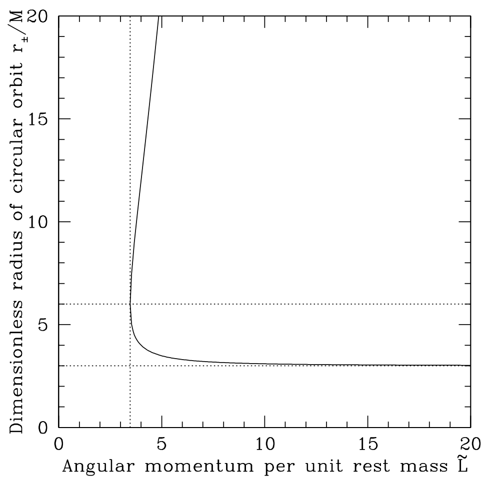
- To build intuition for Schwarzschild orbits and the effective potential, I highly recommend the interactive simulator at `http://www.fourmilab.ch/gravitation/orbits/`. Note that it crashes and requires reloading if you accidentally fall in.

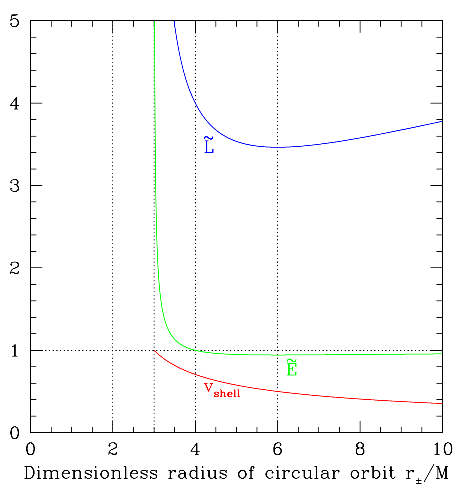
Circular orbits

- Circular orbits are possible at extrema of the effective potential (follows from requiring $dr/d\tau = d^2r/d\tau^2 = 0$), with a minimum giving a stable orbit and a maximum giving an unstable orbit. Setting $\tilde{V}'(r) = 0$ gives the possible circular orbit radii

$$r_{\pm} = \frac{\tilde{L}}{2M}\left(\tilde{L} \pm \sqrt{\tilde{L}^2 - 12M^2}\right).$$

This is plotted in the figure. The larger of these solutions is the stable circular orbit, while the smaller is the unstable orbit at the maximum. If the angular momentum is too small ($|L| < \sqrt{12}M$), no stable orbit exists and the object will either fly off to infinity or be devoured.





- Inverting the last equation gives (see plot)

$$\tilde{L} = \frac{r}{\sqrt{\frac{r}{M} - 3}}.$$

$\tilde{L}$ is infinite both for $r = 3M$ and $r = \infty$, taking its minimum value $\tilde{L} = \sqrt{12}M$ for $r = 6M$, the innermost stable orbit.

Table 1: Interesting of circular orbits

r	v_{shell}	$\tilde{E}$	$\tilde{L}$
$3M$	1	∞	∞
$4M$	$\frac{1}{\sqrt{2}}$	1	$4M$
$6M$	$\frac{1}{2}$	$\sqrt{\frac{8}{9}}$	$\sqrt{12}M$
∞	0	1	∞

- Speed in a circular orbit according to a shell observer (see plot):

$$v_{\text{shell}} = \left(\frac{r}{M} - 2\right)^{-1/2} = \frac{\beta_r \gamma_r}{\sqrt{2}}$$

- Energy per unit rest mass in a circular orbit (see plot):

$$\tilde{E} = \frac{r - 2M}{\sqrt{r(r - 3M)}}$$

  $\tilde{E}(r) = 1$ at $r = 4M$ and $r = \infty$, and takes its minimum value $\tilde{E} = \sqrt{8/9}$ for $r = 6M$.

- The unstable $r = 4M$ orbit is the "tourist orbit", since a rocket full of tourists will have the same energy there as it will far from the black hole, hence making it accessible with a minimal expenditure of rocket fuel. Using no rocket fuel at all, tourists can follow a spiral trajectory winding many times around the black hole to just outside $r = 4M$ and then spiraling out again - or spiraling into the black hole if the calculation was slightly off! In Newtonian gravity, the analogous orbit (with just barely enough energy to escape to infinity) is a parabola, *i.e.*, much less interesting than this nearly infinitely would up spiral. (Amusing factoid: in four or more spatial dimensions, Newtonian gravity becomes more qualitatively like GR, admitting both such a spiral solution and "death orbits" that spiral into the point mass!)

Radial orbits

- Our infalling portable clocks satisfy

$$\frac{dr}{dt_{\rm ff}} = -\beta_r,$$

so integrating

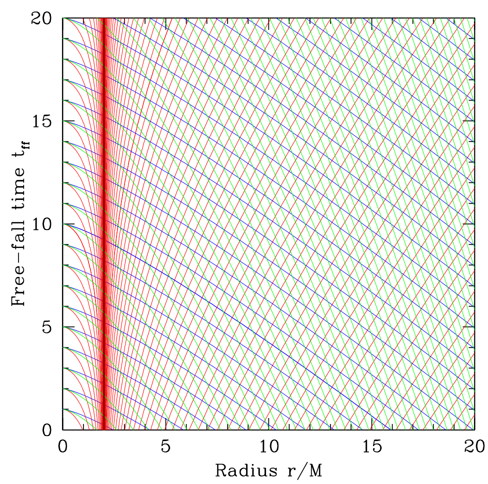
$$dt_{\rm ff} = -\frac{dr}{\beta_r} = -\sqrt{\frac{r}{2M}}\,dr$$

gives

$$t_{\rm ff} = t_{\rm ff}^* - \sqrt{\frac{2}{9M}}\,r^{3/2} = t_{\rm ff}^* - \frac{4M}{3\beta_r^3},$$

$$r = \left(\frac{9M}{2}\right)^{1/3} (t_{\rm ff}^* - t_{\rm ff})^{2/3},$$

where $t_{\rm ff}^*$ is the time when the clock hits the $r = 0$ singularity. The plot shows worldlines (green) of such falling clocks.



- More generally, $\tilde{L} = 0$ for radial orbits, so our radial equation of motion simplifies to

$$\frac{dr}{d\tau} = \pm\sqrt{\beta_r^2 + \tilde{E}^2 - 1}.$$

The falling clock example above was simply the special case $\tilde{E} = 1$, for which $d\tau = dt_{\rm ff}$.

Photon orbits and the deflection of light

- The orbit equations above aren't useful for photons since they involve $d\tau$, which is zero for photons.
- Taylor & Wheeler show that eliminating $d\tau$ between the equations and taking the limit where the rest mass $m \to 0$ gives

$$\begin{aligned} \left(\frac{dr}{dt}\right)^2 &= \gamma_r^{-4}\left[1-\left(\frac{b}{\gamma_r r}\right)^2\right] \\ r\frac{d\varphi}{dt} &= \pm\frac{b}{r\gamma_r^2}, \end{aligned} \tag{4}$$

  where the *impact parameter*

$$b \equiv \frac{L}{E}$$

  is the only constant of motion that we need to keep track of.

- The impact parameter is related to r_{min}, the distance of closest approach, by

$$b = \gamma_{r_{\text{min}}} r_{\text{min}}.$$

  Since this shows that $b = r_{\text{min}}$ if $M = 0$, we can interpret the impact parameter b as the smallest r-coordinate that the photon would ever get if the black hole were not there and the photon simply moved in a straight line.

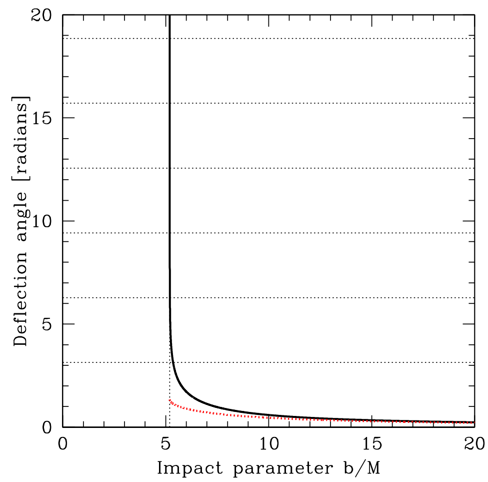
- For a photon, the only closed orbit is a circular one with $r = 3M$.
- **Deflection angle**: Consider a photon arriving from far away, getting deflected by the gravity of a star or a Schwarzschild black hole and flying off to infinity again. As shown in Taylor & Wheeler project D, the total deflection angle is

$$\Delta\phi = -\pi + 2\int_0^1 \left[1-u^2-\frac{2M}{r_{\text{min}}}(1-u^3)\right]^{-1/2} du,$$

  where r_{min} is the distance of closest approach, related to the impact parameter b is given by

$$b = \gamma_{r_{\text{min}}} r_{\text{min}} = \left(1-\frac{2M}{r_{\text{min}}}\right)^{-1/2} r_{\text{min}}$$

  In the plot below, I've done this integral numerically.



Interesting special cases:

- $\Delta\phi \approx \frac{4M}{b}$ for $b \gg M$ (dotted red curve in the plot). Einstein applied this approximation to the deflection of starlight near the Sun as confirmed by Eddington in 1919 and later to exquisite 0.1% precision for radio waves from the quasar 3C273 passing near the Sun. A heuristic Newtonian estimate gives only $\Delta\phi \approx \frac{2M}{b}$.
- $\Delta\phi \to \infty$ as $b \to \sqrt{27}M \approx 5.196M$, corresponding to the photon getting captured and making infinitely many orbits as it spirals in toward the circular $r = 3M$ orbit.
- For $b < \sqrt{27}$, the photon disappears into the black hole. This also means that no photons can come from the black hole direction towards you with larger impact parameters, so that when far from a black hole, you will see it as a black disc of radius $\sqrt{27}M$, *not* $2M$. In other words, the black disk appears with $27/4 \approx 7$ larger area than you might naively expect.
- The plot shows that for $b \approx 5.357M$, $\Delta\phi = \pi$ (dotted horizontal lines show π, 2π, 3π, *etc.*), which means that you can see your own image in a circle around the black hole of thie radius.
- The plot also shows that there will be an infinite number of smaler circular images of you, piling up towards the innermost radius $r = \sqrt{27}M$, so pointing your telescope at the perimeter of that scary-looking black disk reveals quite an interesting "halo".

The field equations (optional)

The *Einstein field equations* are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where

$$\begin{aligned}
G_{\mu\nu} &\equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \\
R &\equiv g^{\mu\nu} R_{\mu\nu}, \\
R_{\mu\nu} &\equiv R^{\alpha}_{\mu\alpha\nu}, \\
R^{\alpha}_{\mu\nu\beta} &= \Gamma^{\alpha}_{\nu\beta,\mu} - \Gamma^{\alpha}_{\mu\beta,\nu} + \Gamma^{\gamma}_{\mu\beta}\Gamma^{\alpha}_{\nu\gamma} - \Gamma^{\gamma}_{\nu\beta}\Gamma^{\alpha}_{\mu\gamma}, \\
\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2} g^{\alpha\sigma} \left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma} \right),
\end{aligned}$$

and $g^{\mu\nu}$ is the matrix inverse of $g_{\mu\nu}$, *i.e.*

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}_{\nu}.$$

Here repeated indices are to be summed over from 0 to 3, commas denote derivatives, and G is Newton's gravitational constant. Throughout this section, we will use units where the speed of light $c = 1$. In the Einstein field equations, the dependent variables are the two tensors $g_{\mu\nu}$ and $T_{\mu\nu}$. They are both symmetric, and thus contain ten independent components each. g is the *metric tensor*, and describes the structure of spacetime at each spacetime point x^{μ}. $T_{\mu\nu}$ is called the *stress-energy tensor*, and describes the state of the matter (what is *in* space) at each point. The quantities $G_{\mu\nu}$, $\Gamma^{\alpha}_{\mu\nu}$, $R^{\alpha}_{\mu\nu\beta}$ and $R_{\mu\nu}$ are named after Einstein, Christoffel, Riemann and Ricci, respectively.

The Schwarzschild metric is obtained by setting $T_{\mu\nu} = 0$ except at the origin and solving for the most general spherically symmetric time-independent metric. Requiring this requiring only azimuthal symmetric (ϕ-independence) gives the Kerr metric corresponding to rotating black holes. For the FRW metric, $T_{\mu\nu}$ is not zero but takes a simple diagonal form independent of position.