YEN-JIE LEE: So let's get started. This is our goal for 8.03. So you can see, during the exam number one, we have covered the first half of the goal, and we are actually making progress to learn about boundary conditions in one dimensional system and also in two dimensional system today. And we will actually talk about phenomena related to electromagnetic waves and optics today, which we will be able to learn two very important fundamental laws related to geometrical optics. OK, so that's the excitement.

And then we started a discussion of two dimensional or three dimensional wave last time. And just in case you haven't realized that, there are two ways to go to higher dimension. So the first way is to increase the number of objects and place that in two dimensional or three dimensional space. And that is the kind of things which you will discuss today. So for example, I can have particles arranged in two dimensions which form membranes. And then we can also, on the other hand, change the direction of the electromagnetic wave, for example, as a function of time, and that's another way to go to a higher dimension. And today, as I mentioned before, we are going to talk about the first case, and on Thursday we are going to talk about the second way to go to higher dimension, which is related to polarization, et cetera.

In general, higher order dimensions are hopeless. They are super complicated. And, in general, we don't really know how to solve this kind of system. Fortunately, in 8.03, what we have been doing is focusing on a small subset of questions of which are actually highly symmetric. Therefore, we can actually solve it analytically. So that will be the focus of 8.03, so that we can actually learn some physics intuition out of this kind highly idealized system. And the system which we are going to focus on today is shown here. It's a two dimensional system, which you have array of masses placing the x and y, x, y plan. And that is the system we are going to solve today. And we will learn a lot of interesting phenomena coming from the solution of this kind of system.

Before we start a discussion of two dimensional system, I would like to remind you of what we have already learned from lecture eight. So that was about a system which consists of infinite number of mass and the infinite number of strings, and each string have string tension T. And all the mass, when they are in equilibrium position, the distance between all those mass in the x direction is a, OK? So we have solved this system before with space translation symmetry. And this is just a reminder that the dispersion relation, which we got a lot time, omega, as a function of k, is t over ma sine Ka over 2. So that was just a reminder of what we have learned from lecture eight.

So by now you should realize that, OK, dispersion relation is unusual. This is actually telling you that this is a dispersive media, right? Because if you calculate the ratio of omega and k, you'll see that this is actually not a constant. So after all the discussion from previous lecture, you should be able to immediately realize that. And any wave propagating on this kind of system, there will be a dispersion phenomena happening in this kind of system. OK? And also from the previous lecture, we'll have learned that the eigenvectors based on space translation symmetry, it's exponential of ikx, where x is defined as j times a, where it's a is a label to tell you what which mass I was talking about.

Now today, we are going to extend this to a two dimensional system. So instead of a one dimensional system we have a two dimensional array. So all the little mass all have mass equal to n, and they are placing xy plan. The coordinate system, which I defined is here. x is horizontal, and the y is vertical, and the z is actually pointing to you. And all those little mass can only oscillate toward you or going away from you, so in the z direction. It can only oscillate up and down in the z direction. And in this system we have the length scale, which is the horizontal distance between mass, is called aH. And in the vertical direction, the scale of the distance between mass is av. Also, we have string tension-- two different kinds of string tension for the vertical and horizontal direction. The vertical direction, you have string tension Tv, and in the horizontal direction you have string tension Th. OK

So how do we actually describe this kind of system, right? The first thing, as we did before, is to label those little mass by my label. And my label is called Jx and Jy, which tells you which mass I was talking about in this system. Once I have defined that, the labels, I will be able to write the position of all those mass, the x direction position and y direction position, in terms of J and the A. So for example, x position of 1 over the mass will be written as Jx times ah. And y position of a specific mass, you can write it down in terms of a Jy times av. So all those things should be pretty straight forward.

The interesting part is that, as we identified in the last lecture, this system is highly symmetric.

It has space translation symmetry, right? Therefore, we can actually immediately figure out what will be the eigenvectors for this system. So the eigenvector-- very similar to what has been discussed here, where you have a one dimensional space translation symmetric system. Exponential of ikx was the eigenvector. Now you have eigenvector which is in two dimension, because you would like to describe not only the x direction but also y direction. And the eigenvector have exactly the same functional form because of space translational symmetry, and it is like exponentional of ixk times x. Multiply that by another exponential function-- exponential iky times y. So I think, until now, nothing should surprise you because this is what we have learned from the one dimensional system analysis.

Based on what we have learned before, we can also immediately write down what would be the dispersion relation. Since we are always considering very small vibration, and this formula is still applies, therefore you can actually write down the dispersion relation-- omega squared k will be equal to 4 Th over Mah sine square Kx Ah, divided by 2 plus 4 Tv divided by Mav sine square Ky Av divided by 2. So this is actually pretty straightforward. And you can see that omega is a function of both Kx and the Ky.

From the eigenvector we can also write down what would be the possible Psi xy. Now the Psi is actually the displacement in the z direction, with respect to the equilibrium position. And that is actually proportional to the eigenvector. So basically it's going to be a exponential ikx times x exponential iky times y. And of course I can write these two terms together, right? So basically what I would get is a exponential i, k is a vector times r, which is a vector. So basically k contains two components, Kx and Ky, and r have also two components, which is x and y.

Again, we see that this is actually a non dispersive medium. And what we are going to do is to make linear combination of all those eigenvectors and figure out what would be the behavior when this system is oscillating at a specific frequency omega, and that is actually the corresponding normal mode at the angular frequency omega. So that is actually pretty similar to what we have done for one dimensional system.

So this is a two dimensional system. Just a reminder about one dimensional system for a while. So there are two eigenvectors which have identical omega, right? So the first one is exponential ikx, and the second one is exponential of minus ikx. What we have done before is to do a linear combination of the two exponential functions, right? So what we can do is that-- OK, now I can create cosine Kx. This is actually 1 over 2 exponential ikx plus exponential minus ikx. Or I can also create sine Kx. And this is actually 1 over 2i, exponential ikx minus exponential minus ikx, for this one dimensional system.

So that's how we figure out when the system is doing one of the normal mode. The shape of the system is like a cosine or a sine. Or in general, you can add these two together, and in general it can be something like cosine Kx plus 5, but 5 is actually some phase angle, which you can figure out by boundary condition. But before we introduce any boundary condition, all the k values, all the five values are allowed. Just a reminder about what we have learned before. So the situation is pretty simple. You have just plus and minus k, and then you make linear combination of these two, then you know what will be the shape of the normal mode.

On the other hand, we are now talking about two dimensional case. So let's take a look at this dispersion relation. The dispersion relation we will have here, omega is a function of Kx, is a function of Ky as well. OK, what does that mean? That means I can have multiple choice of Kx and Ky, which they all produce the same omega value. So it's not as simple as this one any more, as you can see, right? Because when I slightly increase Kx, what I could do is-- OK, I can slightly reduce Ky to compensate the difference. Therefore, I can still keep omega, which is the angular frequency of the oscillation the same.

OK, so that can be seen from this demonstration on the slide. You can see that this is actually one example dispersion relation. This is actually the formula which we have on the board. And what about if I set all those parameters and get, example, omega squared equal to 5 sine squared Kx, and the plus 5 sine squared Ky? OK, so what will happen? If I set my aht and the m value, so that I have this example. What will happen if I have that dispersion relation?

So if I go ahead, and then plot allowed Kx and Ky value, which gives 1, you see a very beautiful pattern, right, on this. So you can see that ha, all those things on the circle can produce angular frequency omega equal to 1. And 1 will be their-- 1 will be the-- normal mode will be all possible linear combination of all those possible Kx, Ky pairs. You have a question?

AUDIENCE: [INAUDIBLE]

YEN-JIE LEE: In general, I think this-- you mean a circular shape?

AUDIENCE: [INAUDIBLE]

YEN-JIE LEE: I think, in general, yes, you do have determinacy because you can-- but the shape will not be circular, for example. OK, so it can be a general function which is like the formula above, but this argument still applies. So if you have some intermediate omega value, you can always slightly increase Kx and slightly decrease Ky, and that will still satisfy the same omega value. Therefore, all the normal modes, we set specific omega will be a linear combination of all those possible normal mode. All those possible Kx and Ky pairs if you have an infinitely long system. And the law also applies to the other example, when I have omega equal to 5, then you have slightly different behavior. But the take home message is that there are many, many pairs of Kx and Ky, which can create the same amount of omega. So that makes things pretty complicated. Potentially, we can always still try to understand this by investigating all the possible k pairs of Kx and Ky.

On the other hand-- this is what we have discussed before. Before you introduce boundary condition for the one dimensional system, there are infinite number of possible k value, right? All the possible k values are allowed. But after you add boundary condition-- for example, I add walls around this system, so that I basically have a fixed boundary condition. So basically, the boundary condition is that the amplitude at x equal to 0, y equal to 0, or x equal to 5ah or y equal to 4av. At the boundary, the amplitude has to be equal to 0, because it is attached to a wall. OK, when this happens this means that we will have have four wall which will have a corresponding boundary condition. So that means I have to satisfy this four boundary consideration of side 0,y evaluated at any time will be equal to Psi Lh, y, t, will be equal to Psi x,0,t, will be equal to Psi x,Lv,t. And this is all equal to 0. So those are not very difficult to understand. Those are just the four walls around the system.

Once you have all those conditions-- and of course I define Lh will be equal to 5 times Ah, because there are 5 strings between the two walls in the horizontal direction. And of course I also have defined here, Lv will be 4 times equal to av. So once I have all those four boundary conditions in place that means I cannot arbitrarily choose k value and the fact, right? Otherwise, I will not be able to satisfy these four boundary conditions. So now we actually will be able to figure out that there will be only four modes in this two dimensional problem, which will give the same omega.

What are the four possible nodes-- what are the four possible eigenvectors? Those are a exponential plus or minus ikx times x, exponential plus minus iky times y, where the Kx-- because of the boundary condition, which we have solved in the one dimensional system-- Kx will be equal to Nx pi divided by Lh-- in order to match the boundary condition, add x equal to 0 and x equal to Lh. And Ky will be equal to y times pi divided by Lv. That's actually designed to match the boundary condition at y equal to 0 and then y equal to Lv.

So you can see that, like when we've seen before with one dimensional system, after you introduced the boundary condition it's not an infinity long system anymore that allowed k value, which is the length number in the x and y direction. For example, in this case, it's also become limited, and only a limited number of possible values are allowed. In this case, Nx is allowed to be equal to 1, 2 until 4, and Ny is equal 1,2,3 in this system we are talking about. Any questions so far? Yep?

AUDIENCE: I think you mentioned that Kx and Ky are directly related rather than inversely related, but I'm sort of confused as to why that is. Because if you want to maintain the frequency, it increases the wave numer and [INAUDIBLE].

YEN-JIE LEE: Yeah, so I was talking about when I choose Kx and Ky in the infinitely long system. OK, all of the possible values of Kx and Ky are allowed because I have a infinitely long system with no boundary condition. And in that case, going back to this dispersion relation, I have the freedom to-- OK, so when I increase a little bit, Kx, I can always decrease a little bit, the Ky. OK, so the question is why that's not the case, right, for the discrete case. As you can see from here, after we introduced the boundary condition, the four boundary conditions especially describe the boundary of the four walls. And what is going to happen is that you will also see that the allowed Kx value is becoming limited, because you cannot arbitrarily choose with lengths, right, if you choose a side along the wavelengths like what we have been trying to do for the infinitely long system-- not that it matched the boundary condition. Therefore, you don't have this degree of freedom to choose slightly higher or slightly lower Ky when I change a Kx. So you can see that the allowed value are discrete. Therefore, the number of possible combinations of Kx and Ky is also limited. And in this case, it's actually very likely to be limited to be only four pairs, which is actually plus, minus Kx and the plus, minus Ky. All right. Thank you for the question.

OK, then once I have those I can do a linear combination of these four possible eigenvectors. And also, at the same time, I will try to match the boundary condition. So if I jump forward, basically what you can conclude is that Psi Nx and y, so that's with an Nx value y value chosen for the determination of Kx and Ky. And is this actually a function of x and y and of course also time, when I also make it oscillate as a function of time. This will be equal to some arbitrary constant, A of amplitude Nx, Ny sine Nx pi x, divided by Lh sine Ny times y divided by Lv. And of course, you can see that this is actually sine, right? It's actually, the same as what we have done for the one dimensional system, right?

So if you have two boundary conditions that said, look, the beginning and the end, therefor, the corresponding normal mode is always a sine function. So that's what we have learned from the one dimensional system. And this is also the case for the two dimensional system. And of course, don't forget this wave function is changing as a function of time oscillating up and down harmonically. Therefore, you have sine omega Nx, Ny times T plus Theta, which is a phase to be determined by initial conditions.

And you can see that the whole equation, a sine sine is multiplied by a sine omega T plus 5 because of beta function, right? So that means the shape is actually going up and down harmonically. So the shape is fixed, which is sine times sine, and the whole thing is oscillating at the same frequency at the same phase, which is the definition of normal mode, right? Just a reminder. And how do we actually imagine what is actually happening? That brings me to the demonstration, so we can really visualize how this kind of system will look like by a little simulation.

So, suppose I choose Nx and Ny equal to 1 and see what will happen. This is the kind of oscillation you will expect, right? So if you choose Nx equal to 1, Ny equal to 1, then this is a system. Basically you have sine function with no node in x and y direction. Therefore, if you do get this simulation, you can see that there will be no node in the x,y plane, and all those particles are either going toward you or going away from you. They only oscillate in the z direction in this simulation. And also, you can see that now I can increase, for example-- I can increase the Kx by setting Nx to be 2 and see what will happen.

So what is going to happen is that if I have higher Kx in the x direction-- so the next possible normal mode is that you have a full sine wave in the x direction, then you are going to see two components in this demonstration. And one part of the system is actually moving toward you while the other half part of the system is actually moving away from you. And you can actually see the node, or nodal line in this case, because we are talking about a two dimensional system in the middle of the distribution. Of course, we can always go crazy, right? I can set this to a really high value. So in this case, the highest value I can set these is 3 and 4, and see what happens. And this is actually a beautiful shape which is actually complicated but understandable, as you see in this demonstration. And all those little particles in this system are oscillating up and down at the same angular frequency and also at the same phase. Any questions?

OK, so now we have done the discrete case, right? And of course we can also go to the continuous case. So if we go to a continuous limit, now I can assume that there is a symmetry between a horizontal direction and the vertical direction. I assume that Th is equal to Tv is equal to T. And also I assume that the length scale in the x direction and the y directions is equal, and the length scale is A. In order to make the whole system continuous, I need to increase the number of objects in the system, and at the same time I also need to decrease the distance between all those objects. So therefore, I need to have-- this length scale goes to 0. And what is going to happen is that if I rewrite my omega, which is a dispersion relation, what I am going to get is 4T divided by Na Kx squared, A squared, divided by 4, plus 4T over Na, Ky squared a squared over 4. This is issue because I am taking-- Ah and V need to be equal to-- and also having to be a very, very small value. Therefore, sine theta is roughly theta, right? Therefore, I can immediately write down this expression. And this will be equal to Ta divided by N, Kx squared plus Ky squared.

So we are facing exactly the same situation. When I decrease A, I am going to add more objects into the system, but I don't want to have an infinitely large mass. Therefore, I also need to ensure the fix the ratio of m and a, so that when I actually increase the number of objects, I don't actually make the total mass go to infinity. So what I could do is I can define Rho S is actually the surface mass density. So the surface mass density is defined as m divided by a squared. And I can also define a surface tension. Surface tension Ts will be equal to T over a. And in this case, basically, I will be able to control, so that when I increase the number of objects, mass doesn't go to infinity, and I have constant surface tension and constant surface mass density. If I have defined this to quantity then this will become Ts divided by Rho S Kx squared plus Ky squared, and this will be equal to Ts divided by Rho S, k vector squared. And this k vector is a two dimensional vector.

So we are actually almost there to make it continuous. So now I can make a goes to a very small value. We fixed the Ts and the row S. Very similar to what we have learned from the one dimensional case. Basically what we actually found is that time in the one dimensional case is that M minus 1 K metrics become minus T over Rho L, partial squared, partial x squared in the one dimensional case. And in the two dimensional case, without working through all the detail of mass, basically what we are going to get is partial square partial T square Psi xy-- It's actually a function of x and y and the time, right, because this is actually a two dimensional system. And this will be equal to V squared, partial squared, partial x squared plus partial squared, partial y squared, Psi xy and T-- very similar to what we have done for the one dimensional system.

And of course I can, as you define this, as del squared. And basically what you are going to get is V squared, del squared, Psi x,y,t. So basically we again see this wave equation, but this wave equation is now a two dimensional wave equation. And we can also figure out what will be the V value, right, so what will be the velocity? The velocity which is going to be square root of Ts over Rho S. This is very similar what we have done for the continuous case, and in this case, what replaced T over Rho L is Ts over Rho S. Therefore, what we actually see that if I increase the surface tension, then the velocity will increase. If I decrease the mass per unit area, Rho S, then I will be able to have a much faster traveling wave from this kind of media. And what we can actually immediately also write down is that the Psi will be proportional to A sine Kx times x, sine Ky times y, and Psi omega T plus 5, where omega is calculated from the input Kx and Ky for this standing wave solution.

And very similarly, I can also argue that-- in the three dimensional case I can actually follow exactly the same argument. Basically, in the three dimensional case, as well, we already see in the electromagnetic wave discussion, the three dimensional wave equation can be written as partial squared, partial T squared Psi is a function of x, y, and z and T. And this will be equal to V squared, partial squared, partial x squared plus partial squared, partial y squared plus partial squared, partial V squared-- Psi is a function of x,y,z, and T. Any questions so far? Nope? OK, so everything is crystal clear, right?

OK, so this is actually the animation, which I showed you before already. So this is actually the two dimensional vibration of membranous. So basically the first one is what I have shown you when I choose a very small K value, which only make half of the sign and which match the boundary condition. Basically you see that there are oscillation, which you have the middle part, which is either going toward you or going away from you in this continuous system. So basically the solution is actually remarkably the same as what we have seen in the discrete system. OK, that's actually what I wanted to say. And also, of course, you can increase the k value, so that you go to the higher frequency normal mode. And you can see that if you have more and more nodal lines, which is actually the lines describing the-- the lines which you actually have no oscillation at all on the surface.

For example, in this case, the nodal lines, as you're passing through the middle of this figure-- because all those little mass, all the other high particles are vibrating like crazy. But all the particles on this line, the nodal line, they're not at all moving, because that's actually at this position-- which is having one of the sine function equal to 0. Therefore, no matter what you do as a function of time, how you evolve the system, all those particle at that line will not move at all.

And this was demonstrated from this table here. It's actually Chladni figures. You can see that in a two dimensional case the figures can look very complicated. So basically what it's showing here is that you have a square plate and it's attached to a vibrator, and basically this vibrator can be controlled. I can change the frequency of that vibration. When I reach resonance, which excites one of the normal mode, then this plate will be oscillating in a specific pattern. And those lines are actually showing you that the plates, which you have no oscillation at all as a function of time. Because if I, for example, turn on this demo again, you can see that if I turn on this demo-- you can see that all the sand on the plates are vibrating because now I am oscillating this plate by the vibration generator and the button-- by the motor and the button.

And if I change the oscillation frequency so you can see that this frequency doesn't match with one of the normal mode frequency. Therefore, they will not be a lot of activity. But if I now change the frequency, so that it matches with one of the oscillation frequency for one of the normal mode of this system, you can see the, oh, some really cryptic pattern is formed! You can see that, oh, it have a very complicated pattern. And if I put my finger in one of the lines here I don't feel the vibration, but on the other hand, if I put my finger here, I can actually feel that there's lot of vibration at that point. I can always change the frequency and see what will happen. And then you can see that now I increase the frequency, and now I am actually trying to excite another mode. Now I need some more sand. You can see that I randomly throw sand on this plate, and then you can see that those centered as you bounce it around until it sits on the nodal line, which no vibration actually happens.

OK, so let's go back to one of the lower frequency mode, which we showed you before. Now the question is, OK, you can see this complicated pattern almost look ridiculous, can we actually reproduce this pattern by our calculation? So we have seen that, OK, I can conclude that the normal mode looks like this, right? So therefore, I must be able to explain all those patterns, which is actually shown in this experiment. So that's actually what I am going to do to give you a try.

So this is a little demonstration which I actually wrote. This demonstration actually has the solution to this two dimensional problem. And also the boundary condition is that-- or say, the condition which, as you can see on my solution, is that I require the center of the plate to be driven, because that's where I start to vibrate this plate. And I drive this plate up and down to see what is going to happen. So from this analytical calculation you can see that you expect a circle in the middle and also four lines which actually cover this circle. And also there are some strange structure at the edge of the plate.

And you can actually compare this calculation to this result. It doesn't really match perfectly. So you can see that there is some imperfection, but you get that ring in the middle, and you do see these 1-2-3-4-5-6-7-8, 8 lines produced in this experiment. So this experiment is not perfect because there's a, you know, stiffness of this thing and also some energy dissipation, et cetera. But you can see that, sort of, we can actually use our calculation to explain this pattern! That's really cool, right? And the advantage is that now I have this wonderful simulation. I can put in all the crazy numbers, and you see that, huh, if I increase the K value, I can really make all kinds of ridiculous patterns out of this. And all these things can be kind of realized by this experiment.

So you can see that, for example, I can now also turn on this, and I can actually increase the frequency to a very high frequency, for example. Then I can see that, oh, the pattern really becomes much more complicated. Now I have a circle and there are many, many more structures which you're seeing in the surrounding area. And of course I can again increase, increase, and see what happens. I don't know what is going to happen because every time I do this experiment I get a different pattern. OK, now this seems to be a very nice frequency. It's getting harder and harder. You can see that this is really crazy. Holy mackerel, right? What the hell is this? So you can see that all those crazy patterns can be created. And of course, during the break, you are welcome to come forward and play with this.

So you can see that we can actually understand, sort of, the pattern produced from this experiment. That's actually very exciting, because that's actually why we are physicists, right? We would like to know why those patterns are formed, and now you know why. Those patterns are formed because there are nodal lines in this two dimensional normal mode modes. And the little sands really love to sit there, because you want to sit in a place which you don't have a lot of vibration. It's not very comfortable, right? So you sit in the place, which, hm, vibrates? Your problem. Vibrate's your problem. I sit here where there is no vibration. So that's basically how we explain these strange figures which we can see. And just for fun you can see that I can also generate all kinds of craziness. You can input all kinds of different Nx and Ny values, and you get all those wonderful figures for free. Maybe we can actually make some T-shirts with all those figures on the T-shirt, right?

OK, so we had a lot of fun with this two dimensional plate. How about what will happen if I have a circular plate? What does it do? Unfortunately, I would not be able to solve the two dimensional plate problem in front of you because that will give you a Bessel function, which is not the end of the world, but that's actually kind of complicated. If I put it in mid-term exam, that's actually not very encouraging, right? But I can actually tell you what will be the solution. The solution will be a Bessel function.

Basically you will have a lot of ring-like structures if I have a circular plate. And I can actually do an experiment which actually shows you the behavior of the circular boundary condition and see what kind of pattern can we see. So here I have a kind of complicated experiment. So here I have this ring, which I would like to produce some film on this ring. So see if I am successful. Kind of. OK. Now I can put this a soft film in front of the speaker. I can actually oscillate this thing-- membranes by the speaker. Oops, don't want to destroy everything here.

All right, so now I can turn on this, so that we have light. And of course I will turn on the signal generator, so that I can hear-- I can actually start to vibrate the membranes. Before I do that, I have to turn everything off, hide images. All right, I hope you can see something. Can you? Can you see something on the-- it's kind of difficult to see it, but that should be there. OK, now I can turn on this speaker, and you can see that there are some patterns which it's probably difficult to see. Kind of see, right? There are rings. You can see it on the speaker. So you can see that now I have one, two, three-- three rings, right? Because I couldn't turn off the light, which is actually emitted from the sun, right? So I cannot turn off sun. Therefore, you can barely see this figure. So we shall explain the result of this experiment.

And you can see that if I increase the frequency, according to the solution from Bessel function, you will see more rings got excited. So you can see that now I have one, two, three, four-- four rings. And of course I can continue to increase, increase, and increase. And you will see that there are even more rings produced. Essentially what I'm doing is actually really trying to vibrate and excite one of the normal mode by this loud speaker. And you can actually kind of see-- I hope you can kind of see it. If you can still see it, that means you need to check your eye because the membranes is broken. OK, so I think you sort of get this idea, and I'm going to turn off this wonderful machine and go back to the lecture.

So this experiment is kind of hard to reproduce in your study room, right? I think everybody will agree. And there's another one which is actually kind of easy to reproduce, which I will encourage you to try it-- so if you have time. So this is from Jake. He sent me this wonderful video when I was teaching the 8.03 class. They found that they could excite two dimensional waves in this way. Can you see it? It's wonderful. You can see there are very high frequency oscillation, which actually excite these two dimensional wave.

And you can see that lots, and lots, and lots of rings are excited. And then you can see very clearly from this simple experiment, what you really need is a cup of water. And you rub it against the surface of a table, then you'll be able to excite all the crazy patterns, which you can actually see from this two dimensional system and with two dimensional boundary conditions. OK, so we will take a five minute break before we enter the next part of the discussion. And we come back at 35.

OK, welcome back, everybody. So what I'm going to do now is to continue the discussion, the one we actually got started, of the two dimensional and three dimensional system. And we have actually studied the behavior of standing wave, or normal mode, for this two dimensional system. And what I am going to do is discuss with you, a two dimensional progressive wave. So I will stick to a really simple example, which are plane waves. OK, so in the case of plane waves, which we discussed when we actually discussed the EM waves, you have the following functional form. Psi is a function of r and the t. And this will be equal to A exponential i. The k is a vector now, and it's pointing to the direction of the propagation of this plan wave. And this k is dot with r vector minus omega T, which is the oscillation frequency-- angular frequency. And evaluated at a specific time. And this is expression actually describes a plane wave where the direction of propagation is described by this k vector. And of course you can actually have the wave front, which is actually the peak position of this plain wave. And the distance between the peak position-- so if you can imagine that this is like this. So if you look at the distance between peak position that will give you the wavelengths, right? The wavelengths, now that will be equal to 2 pi divided by k, right? In this case, it's the length of this k vector. Just a reminder about what we introduced in the previous lecture. And we were using this to describe electromagnetic wave and such a kind of expression can be also be used to describe sound waves and also vibration on the membranes, et cetera, progressive waves.

So if there are no other medium like what we actually have in this slide-- so we have nothing else. I have a membrane with a surface tension Ts, and Rho S is the mass per unit area. Then basically, this progressing wave is going to be traveling at the speed of v, which is equal to square root of Ts over Rho S, and I can actually define that to be some constant c divided by n. So c is some constant, and m is another constant which actually, the ratio c and n is equal to v. And I will need that expression later, only later, not now.

If I have nothing else and that this system actually filled the whole universe, then what is going to happen is that this progressing wave is going to be propagating, propagating, propagating, propagating. Nothing will change until the edge of the universe. It doesn't actually introduce any excitement. So that's what we have already learned from when we have discussed electromagnetic interaction, and now the same expression can also be used for the description of the membranes.

And then now to make this problem more exciting, what I'm going to do is to introduce a boundary. So the boundary is in the middle of this slide. And I will assume that the horizontal direction to be x equal to 0-- the horizontal direction to be in x direction and the boundary is at x equal to 0. And when you pass this boundary, there's another kind of material with surface tension Ts prime and slightly different mass per unit area, your s prime. Based on the expression we got for the velocity we will be able to conclude that v prime will be equal to square root of T prime S divided by Rho S prime. And that will be equal to c over n prime. And c is the same constant which I used for the left hand side system. And n, later, you will realize that that's a refraction index in a discussion.

So the question which I would like to ask is, OK, now I have a prime wave propagating in the first system. And it met a boundary, and the question is what will happen when I have the incident wave coming into the system? So before that, I also need to write down the dispersion relation, right. So dispersion relation can be attempted by plugging in a normal mode Psi function into the wave equation. So what I can immediately obtain is that the dispersion relation, omega squared is equal to V squared, Kx squared, plus Ky squared. You can actually check this expression by plugging in this function into the two dimensional wave equation, and you will get that expression, OK?

And that means omega cannot be arbitrary number. It's as you decided by Kx and the Ky. Or say, if omega is the side and one of the k is the side, then the third number, for example in this case, Kx, is as you decided by the dispersion relation which we have here. So, coming back to the original problem we are posting, I have, now, the incident wave coming into this system. I would like to know what will happen at the boundary when I have two systems with a left hand side propagating at-- the speed of the propagation is v, and right hand side's speed of propagation is v prime. What is going to happen?

Assume my guess that I am going to get a refractive wave and a transmitted wave. So that's based on what we have learned from the one dimensional system. If I call this the left hand side, and call the right hand side system right hand-- the right hand system, r. So I can write down the wave function Psi L describing the left hand side. This will be equal to A exponential of ik dot r minus omega T. This is actually the incident wave-- describing this incident wave. And as you might guess, there should be some kind of refraction, right? So once this wave actually passed through the boundary, or touch the boundary, there should be some kind of refraction, right?

So the refraction, I can actually write it down in this form as sum over alpha, r alpha is actually the coefficient over amplitude as function of the normal modes-- as a function of the progressing wave number, which I have shown. And I can actually sum over all kinds of progressing wave numbers. Exponential ik alpha times r minus omega T. So this is a general form of refracting wave. k alpha is describing the direction of the individual refractive wave, and alpha is labeling the individual refractive wave. But I don't know what will be the functional form for the k alpha for the moment. So therefore, I try to sum over all the possible alpha. And I would like to figure out what will be the allowed alpha by matching the boundary condition. So in short, the right hand side turn essentially is actually describing the refractive wave.

And finally, passing through this boundary condition, let's look at the right hand side. Right hand side, Psi r, is going to be sum over beta on the transmission coefficients tau beta, which is the original amplitude, exponential of i, k beta times r minus omega T. So again I don't know what will be the behavior of the transmitted wave. Therefore, I have summed over all the possible values. And this is actually the functional form for the transmitted wave.

I also know that k alpha vector squared will be equal to omega squared Rho s over Ts, and this will be equal to omega squared v squared, because of the dispersion relation in the left hand side. So basically, if you look at the left hand side dispersion relation, the length squared of this k vector will be equal to omega squared times v squared, right? This is just a dispersion relation of a non-dispersive medium. And also, I can actually figure out what will be the-- allowed length for the k theta. So the k theta squared will be equal to omega squared v prime square, because this progressing wave is actually the transmitted wave, is actually traveling in a second medium.

So look at what we have done here. So we have an incident wave. We will wonder, then, what is going to happen. Our physics intuition tells me that, you must get a refracting wave, oscillation frequency should be the same. Otherwise, as a function over time, you cannot match the left hand and right hand side. And you also get a transmitted wave. But I'm now in trouble because I have so many turns. I'm summing over alpha infinite number of turns, and I don't know what will be the coefficient for the r alpha and the tau beta, which are the transmission coefficient and then refraction coefficients.

So what I need to do, as you might guess, is to use the boundary condition. So now I am writing down, already, the general expression. Now I'm going to use the boundary condition to actually limit the choice of the possible k alpha and the k beta. What is actually the boundary condition? The boundary conditions are that at x equal to 0-- that's actually at the position of this line-- the membranes doesn't break. Otherwise, suddenly the membranes break, and this is the end of the discussion, right? Like, what we have done before, right? So the membranes doesn't break, so that the propagation can continue.

So what does that mean? This means that if I evaluate Psi L and Psi r at x equal to 0, Psi 0, y, t. The left hand side will be equal to A exponential i, Ky times y minus omega T, plus summing over all possible alpha, r alpha, A exponential i, K alpha y times y, minus omega T. This is the incident wave transmitted wave evaluated at the left hand side, which is the upper formula. And that will be equal to the right hand side, which is containing only the transmitted wave. So basically you have summing over beta, tau beta, A exponential i, k beta, y times r minus omega T.

And this expression, this boundary condition, should hold true for all the possible y, right, because the boundary condition is valid at x equal to 0. I didn't specify the value of y. So therefore I can actually put in all the possible-- oh, this should be y. Sorry for that. I can actually vary the y, and I will figure out that, ah, if I have Ky not equal to k alpha y, that means the wavelengths of the refractive wave and the incident wave will be different. If I have Ky not equal to beta y that means the transmitted wavelengths is going to be different from the incident wave. That means, no matter what I do as a boundary of y, the membranes will break. Therefore, in order to make this equation valid, the only choice is that when k alpha y will be equal to k beta y and equal to Ky. So that means the wavelengths projected in the y direction should be equal for the incident wave, transmitted wave, and the refractive wave. Otherwise, as you always move a little bit in the y direction, the membranes will break. So that's actually the condition which you can actually get.

And the interesting thing is that, based on this expression, k alpha, the length of the k alpha, and the length of the k beta is fixed. And I also know what will be the component for the y direction. Therefore, that means the x direction Psi's for that k alpha x and the k beta x are also fixed because of the dispersion relation. So that immediately brings me to this conclusion that basically k alpha x will be equal to minus omega squared over v squared minus Ky squared, and that will be equal to minus Kx. So this is the x component of the refractive wave. And the transmitting wave, k beta x, will be equal to square root of omega squared over v squared minus Ky squared.

If I draw, visualize the relative direction of all the three components, basically, this is essentially the direction of the incident wave, k, and the incident angle is theta. And from this expression, you see that the Ky is equal to k alpha y. Therefore, you have a refractive wave. But actually only the x direction has changed sides. Therefore, you have a refractive wave with exactly the same angle as the incident angle theta. The refraction angle will be zeta as well. And that there will be a transmitted wave with theta prime. And this is essentially the direction of the k prime. And the interesting thing is that the projection toward the y direction, that k prime y, has to be equal to the progression of the original incident wave in the y direction.

So that means I will be able to conclude that-- the y components are the same. Therefore, I can conclude that k sine theta will be equal to k prime sine theta prime. I'm kind of running out of time. And if I define, already as I defined here, velocity is equal to c over n, and the v prime is equal to c over n prime, what I can immediately conclude is that-- give me one more minute-- is that if I have n equal to c over v and the n prime is equal to c over v prime, I can conclude that sine theta will be equal to n prime sine theta prime. Does this look familiar to you? This is essentially Snell's law. How many of you haven't heard about Snell's law. There were a few before. Yeah, OK. No problem at all. Then you learned it.

So that means if I have two kinds of systems in my hand, and I will be able to relate the transmitted wave according to what I have in the incident wave. And you can see that Snell's law-- which were famous for the discussion of optics-- and here, I have no knowledge about optics or electromagnetic waves. So in short, what I want to tell you is that, we have just proved two of the most important laws of the geometrical optics, the refraction angle is equal to incident angle and the Snell's law without using any information about the dynamics. That means all those laws are coming from purely boundary condition and the waves. Therefore, you will expect that this will work for water wave, sound wave, electromagnetic wave, et cetera. O So we will continue the discussion next time. Thanks for the attention. And if you have any questions, let me know. I will be here.

Hello, everybody. We are going to show you a demonstration, a really nice one. It consists of the following setup. So basically I'm going to place some film here. And then behind that there's a loud speaker, which I use as a signal generator and to actually produce sound wave. And this sound wave is going to oscillate the soft film, and then you are going to see the oscillation, or the normal mode's pattern, on the screen. OK, so that is actually the setup which we can actually demonstrate to you two dimensional normal modes.

So the first thing which I am going to do is to produce a soft film. Now I am going to put it back into this setup here. You should be able to see the pattern on the screen. Then I am going to turn on the sound wave generator. You can see, immediately, that the pattern on the screen changed because of the sound wave trying to oscillate the soft film. You can see it directly from here, but it actually looks much more prominent on the screen. And now what am I going to do is change the frequency of the sound wave. And you can see that I'm changing it to a higher frequency. And you can see that there is a more and more complicated pattern formed on the screen. That is because I'm now exciting higher and higher frequency normal modes.

And you can see that now I can actually continue and increase the frequency. And you can see that-- now we can see that the pattern becomes really, really infinitely complicated. You can see this grid developing. And then you can see that eventually that's basically two sine functions multiply each other. One sine function is in the x direction. The other one is in the y direction-- horizontal and the vertical direction. And you can see a really beautiful pattern forming due to the solution we derived during the lecture. And the higher frequency I go, I can see more and more complicating patterns, many more lines developing on the screen due to oscillation of the soft film.

You can see now we have even more lines. And it's actually getting more and more difficult to see the pattern because now the lines are really close to each other. The nodal line, we can see clearly on the screen. Now I am going back down to a lower frequency, and you can see that at low frequency oscillation, the number of lines is actually smaller, and that is because of the smaller oscillation frequency and the longer wavelengths of the normal modes.