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PROFESSOR YEN-JIE LEE: Welcome back, everybody, to 8.03 Today, we are going to continue the discussion of symmetry matrix, which we started last time. And this is what we have been doing. OK? So the thing which we have been doing is to solve the normal mode frequencies of component systems by looking at equation of motion, MX double dot equal to minus KX. And in the end of the day, what we are doing is really to solve Eigenvalue problem-- M minus one K matrix Eigenvalue problem. And we have been exercising this several times the last few lectures.

And the M minus one K matrix says you're describing how each component in the system interact with each other. OK? So that's actually what we are trying to do, to solve the normal modes. So we have been making progress, and we are increasing the number of coupled oscillators as a function of time. And now we finally arrive at our limit, which is infinite system, right? So basically, what we have been discussing is a special kind of infinite system, which actually satisfies translation symmetry, right? So in general, we don't know how to solve infinite system. If this system is really complicated, no symmetry, then who knows how to solve this, right?

But very luckily, in 8.03 we have started to get a highly symmetrical infinite system, and in this case, it's translation symmetry. And that is really pretty nice, and we can actually use this example to learn an interesting fact, which we can see from the physical system we discuss here. So, one thing which we have been discussing is the space translation symmetry matrix. As a reminder, S matrix, as defined here. Basically, if you have a vector A, which describes the amplitude of an individual component in a system, basically these vectors have Aj, Aj plus one, Aj plus two blah blah blah. All those amplitudes are included in this vector.

And what does this S matrix do? It's actually the following-- so if A prime is equal to S times A, then the base component of A prime will be equal to Aj plus one. OK, so that's actually what this S matrix does to A vector. OK? So you can think about S metrics as an operator. It's actually picking up the Aj plus one component, and moving it to Aj in the new vector. OK? So that's actually what this S matrix does. And we were talking about the Eigenvector of S matrix.

For example, if A is an Eigenvector of S matrix, then we have this relation, SA equal to beta A. Now, beta is actually the Eigenvalue of S matrix.

OK? And then we also discussed last time, that means, based on this logical extension, Aj prime would be equal to Aj plus one based on the definition of S matrix. And now this is going to be equal to beta times Aj, right? Because we assume that that A vector is an Eigenvector of S matrix. OK? That means Aj will be equal to beta to the j A zero. Right? According to this relation. Therefore, we can conclude that Aj will be proportional to beta to the j. OK, beta is still some coefficient, which we have not determined, and it can be anything. OK?

So, in order to consider a system of little masses, which are oscillating up and down instead of going in one direction forever or the amplitude grows exponentially. And we also don't want the system to have the amplitude go to infinity when we go to very very large j value or very small j value, therefore we limit our discussion in the case of beta equal to exponential ika. OK? In this case, the absolute value of beta is one. Right? OK, and that means-- OK, Aj is actually proportional to beta to Aj, right? So if you take a ratio of Aj plus one and the Aj, the ratio is beta.

If beta Is not equal to one or its absolute value is not equal to one, then this Aj value is going to be increasing according to a power law, right? So the amplitude is going to be, whoa, going to a very, very large value, right? And that corresponds to some kind of physical system, but not corresponding to oscillation. OK? And if we do this and assume that beta is equal to exponential ika. OK, if we do this, what is going to happen is the following. So again, the ratio of Aj plus one and Aj is a fixed value beta, equal to exponential ika. But what it does is, instead of changing the amplitude, it's actually doing a rotation in the complex plane. OK? So if this is Aj, J plus one, J plus two, et cetera, as a function of say j, the variable, then what it does is that this operation multiplied by beta is really a rotation in the complex plane.

And while we actually see in the physical system, it's actually a projection of this complex imaginary plane to the real axis. And that would give you a sine function and cosine function. So very interestingly, if we choose wisely, the beta to have absolute value of one, then that will give you a system which is actually oscillating up and down and the amplitude is actually confined within some value. OK, so that is actually quite interesting. The other thing which I would like to talk about is-- OK, I choose to have exponential ika as my beta.

OK, why ka, right? It looks really strange here. Suddenly the k and and the a are coming to play, right? So what is actually the small a? The small a is actually the distance between little mass, just a reminder. So this is actually the length scale between all those little mass, right? Therefore, what I'm doing is to factorize out the length scale, and then we suddenly found that. OK, after I do this, I define beta equal to exponential ika Instead of exponential i theta, right? So you can also do exponential i theta, right? But instead, I gave theta a fancy name, which is k times a. a is actually the distance between mass.

Then something interesting happens, because k suddenly also will have a meaning. It's actually the wave number of the resulting sine wave. OK? So that's actually why we actually factor out this a factor. Then, after that, we actually found out the amplitude would be proportional to exponential ijka, and j is actually just a label of a phase component in the system. OK. So, once we have solved the Eigenvalue problem for the symmetry matrix, S, as we discussed before, if you look at the slides, OK? If S and the M minus one K matrix, they commute. OK? Commutes means that you can actually change S and and the M minus one matrix, you can swap them, OK, when you multiply them together. OK?

If you can swap S and M minus one K matrix, that means they commute. And our conclusion from last time is that they will share the same Eigenvectors. OK? Of course, not necessarily the same Eigenvalue, but they share the same Eigenvector. So that's great news! Because instead of solving M minus one K matrix, which can be really complicated; depends on what kind of physical system you are talking about. I can solve S matrix Eigenvalue problem. OK? And then this Eigenvector, which I just found here, is going to be the Eigenvector of M minus one K matrix.

And that's actually really making things much easier, because now instead of solving Eigenvalue problem of M minus one K matrix, what I am doing is just multiplying M minus one K times A, then you can actually obtain the normal mode frequency omega. OK? So that's the issue of the great excitement. What does that mean? That means, if you have all kinds of systems, which are translation symmetric-- you can have a line, you have however many people together, whatever a system which is so on here. They are all going to have the same Eigenvector.

You see? You already know how they are going to interact with each other, and what is actually the amplitude as a function of j, which is the location label. OK? So that's actually really wonderful. So, in order to help you with understanding of this system some more, we are going to discuss another system which is actually also very interesting. It's actually a spring--

OK, last time we discussed a spring and mass system, right? And then we solved it together. And this time we are going to solve a system which is made of mass and strings. Ok?

So that may actually copy-- OK, let me actually introduce you to this new system we are going to talk about today. It has many little mass here, from left hand side of the universe and right hand side of the universe. Take forever to actually construct this system. OK? Then my student actually carefully link them by strings, and we make sure that the string tension, OK, is actually a fixed value, which is T. OK. T is actually the string tension. And of course, as what we discussed before, the space or the distance between little mass is actually A. OK? So that's, again, the same length scale, which we were using.

And finally, in order to describe all those little masses in the system, I label them as j, j plus one, j plus two, et cetera, et cetera. OK. So, the question we are asking is, what would be the resulting motion of this system, right? So what we can do is what we have done last time, right? We take jth object in this system, and we look at the force diagram and this M minus one-- to write down the equational motion and also the and M minus one K metrics. So what is actually the force diagram of j's component?

OK. If I take this-- so first, I take my jth mass, and it's connected to two strings, right? So there are two strings connected to jth mass. OK? And of course, left hand side you have another mass, right hand side you have another option. OK? And I know the tension of this string. Its actually a fixed value, and the string tension is fixed at the T. OK? So, in order to describe this system, I need to find my coordinate system, right? As usual. So, what is actually the coordinate system I'm going to use.? So I need to define horizontal direction to be x, and the vertical direction to be y.

Therefore, I cannot express this little mass to be-- the position of jth little mass as Xj and the Yj. OK? We can do the same thing for the left-hand side mass is Xa minus one, Yj minus one, and Xj plus one Yj plus one. So to simplify the discussion, what I'm going to assume is that all those little masses can only move up and down, OK, instead of back and forth. OK, so only one direction is allowed, and also I assume that the up and down motion of all those little masses is really really very small. So I can use small angle approximation.

OK, so therefore I can-- this is essentially the zero of Y axis, when I actually-- before I actually move-- wind up with the mass away from the equilibrium position. So when the string mass system is at rest, OK, not really moving, then all the mass are at Y equal to zero. OK? So now, I actually move this mass to Yj, OK? Then, apparently there are two forces acting on this mass. The left hand side is string force, and the right hand side is string force. And the magnitude of the force is actually T. OK? So, in order to help us with solving this problem. I would define two angles-- the left hand side angle to be theta one, and the right hand side angle to be theta two.

Then I can now write down the equation of motion in the horizontal direction, and then in the vertical direction. OK? All right. Since I have assumed that all of those masses can only move up and down, now I mean-- and also the displacement with respect to the equilibrium position, which is Y equal to zero, is really small, compared to, for example, the length scale a. OK? So therefore, I can write that condition explicitly. So I have a condition. I assume that Yj is actually much, much smaller than a, which is the distance between those masses.

And that means theta one and theta two are going to be much, much smaller than one. OK? So that's actually given us a chance to use small angle approximation. So based on this force diagram, I can now write down the two equations of motion; one in the horizontal direction, the other one in the vertical direction. So what I'm going to get in the horizontal direction is M X j double dot. These will be equal to-- OK, there are two forces. Right? Horizontal direction, I would need to calculate the projection to the X direction, therefore the left hand side force will give you minus T cosine theta one, and the right hand side of this string force is going to give you plus T cosine theta two. OK?

And then in the vertical direction, what I'm going to get is Myj double dot. This is equal to minus T sine theta y. Now I'm doing the projection in the y axis in the vertical direction, and minus T sine theta two. OK. And since I have this condition, all the mass can only move up and down, and also the displacement is much, much smaller than a; therefore, I have the small angle approximation. Cosine theta is roughly equal to one, and the sine theta is roughly equal to theta. OK? And I would call this the equation number one, and the second equation in the vertical direction to be equation number two.

OK. So, up to here, everything is essentially exact, and now I would like to make a small angle approximation and see what will happen. And now equation number one will be called MXa double dot equal to minus T plus T, and this is equal to zero. OK, so that means we will not have horizontal direction acceleration. Therefore, in the horizontal direction, there will be no acceleration and therefore no movement in the X direction, or horizontal direction. OK, and now I can take a look at the vertical direction. OK? So basically, what I am going to get is MYj double dot, and this will be equal to minus T. Now, sine theta one will be roughly equal to theta one. So what is actually theta one?

Theta one is going to be-- OK, so this is actually the Yj minus Yj minus one. Right? So now that's actually the difference between the amplitude of the displacement of the mass j, and the displacement of mass j minus one. Right? Divided by a, I get theta one. Right? Therefore, the first sine theta one, will become Yj minus Yj minus one, divided by a. OK? And of course, you can do the same thing for the sine theta two, right? Which is actually just the theta two. Then, basically, what I am going to get is minus Tyj minus Yj plus one divided by a.

OK. Any questions? OK. I hope everybody is following it. OK, so now I can actually simplify equation number two, and basically, what I am going to get is MYj double dot will be equal to minus T over a. Basically, I take T over a out of it again. And I also collect all the terms related to Yj minus one, minus two Yj, plus Yj plus one. OK, I just ask you to rewrite equation number two in a form which we like more. OK, so that is actually the equation of motion, so from now on, I am going to ignore all the motion in the horizontal direction, because in this small angle approximation we have shown you that there will be no acceleration in the x direction, right? So now it's actually getting a step forward again. Basically, we have the equation of motion, and what is usually the next step? The next step is to write down what matrix? Anybody can help me?

STUDENT 1: M minus one k matrix.

PROFESSOR YEN-JIE LEE: M minus one k matrix, right? So actually, as usual, we actually follow the procedure. Now I would like to write down the m minus one k matrix. OK, so before I do that what, I will define-- I will actually assume my normal mode has this functional form, yj is equal to the real part of aj exponential i omega t plus phi. So basically, that tells you that all the components are oscillating at the same frequency, omega, and the same phase, phi. Yes.

STUDENT 2: When there's a scenario--

PROFESSOR YEN-JIE LEE: Yeah?

STUDENT 2: [INAUDIBLE] also [INAUDIBLE].

PROFESSOR YEN- This one?

JIE LEE:

STUDENT 2: No, there should be one [INAUDIBLE].

PROFESSOR YEN-JIE LEE: Here?

STUDENT 2: Yeah.

PROFESSOR YEN-JIE LEE: Ah. OK, maybe I made a mistake somewhere. Yeah I think it should be plus, right? OK. All right. Thank you very much. So now we have all the ingredients and we assume that it has a normal mode of aj, yj in this functional form. And then now I can-- the next step is to get m minus one k matrix. All right. So what is actually m metrics? M matrix is really really straightforward. It's m, m, m in the diagonal terms, and all the rest of the terms are zero. All right. And those, of course, you can also write down k matrix, right?

So the k matrix will be equal to-- there are many terms, and in the middle you have minus t over a, two t over a, minus t over a, and all the rest of the terms are zero. And of course these patterns go on and on. Minus t over a, two t over A, minus t over a and zeros, et cetera et cetera. And this pattern is going to go on forever, because this is actually infinitly long matrix, infinite times infinitely long matrix. OK, so once we have this, we can now write down the m minus one k matrix. What is actually the m minus one k matrix? It has a similar structure to k matrix, right? All those are zeros-- OK, all those are the other values, but it has a fixed structure minus t over ma, two t over ma, minus t over ma, and then zeros.

And this will go on forever in the diagonal term and also the next two diagonal terms. And all the rest of the terms. are zero. OK, Any questions? OK, so that's really nice. Now we have our m minus one k matrix, and the good news is that you don't have to solve m minus one k matrix's Eigenvalue again, right? Because we have solved the Eigenvalue problem of s matrix, therefore what is our left over is to multiply m minus one k by a, right? a is actually one of the Eigenvectors of s matrix. OK, so I am going to multiply that for you, and now we calculate m minus one k equal to omega square a, then I can get omega square out of this calculation. OK?

If I again focus on this term. OK, so basically, what I am going to get is, right hand side, I have omega square aj, OK? Now that's actually from the right hand side, OK? And left hand side m minus one k times a, what I'm going to get is t over ma minus aj minus one plus two aj minus aj plus one. All right? Because if you take this term-- this term is actually in the exact diagonal of this m minus one k matrix, therefore this matches with j, and this will match with j minus one; match with j plus one. OK, therefore, if you multiply m minus one k and the a, you get this result. OK?

And we also know that aj is proportional to exponential ijka, right? Therefore I can take aj out of this and basically I get t over ma aj minus exponential minus ika plus two. Because I take aj out of this bracket, OK. And minus exponential ika. OK. Now I actually can cancel aj. Basically, what I get is, omega square will be equal to t over ma two minus exponential ika plus exponential minus ika. And that will be equal to two t over ma. OK. One minus-- OK, so exponential ika plus exponential minus ika, you are going to get two cosine ka, all right? Therefore, you get one minus cosine ka.

OK. I define omega zero to be square root of t over ma, just to make my life easier. OK? Then, what is going to happen is that I will have omega square equal to two omega zero square one minus cosine ka. Any questions? OK, of course if you like, you can also rewrite this as four omega zero square sine square ka divided by two. OK, so if you like. OK, so look at what we have done. We studied a highly symmetric system, which is as you were shown in the slide. OK, basically you satisfy the space translation symmetry. OK?

Now what we have been doing is to derive the equation of motion, make use of the small angle approximation, then you will be able to find that, OK, only the y direction is actually moving as a function of time. Therefore, based on this derivation of m minus one k matrix, I arrive, and also based on the equation of motion, which I derived from the first diagram, I get this m minus one k matrix. And since we know that m minus one k matrix and s matrix will share this Eigenvector, I can multiply m minus one k matrix, and I come back to a. Then I will be able to solve the functional form of omega square, and that is actually given here. Omega square is equal to two omega zero square y minus cosine ka. OK?

And is this actually telling you that omega is a function of k. OK. What is k? k is actually the wave number and omega is actually the angular frequency of the normal modes, right? So that means what we were talking about-- that means if we fix the wavelength or the wave number, k, then there will be a corresponding omega. OK? If you fix the wavelengths you are talking about, then the omega is also fixed by this omega of k function. OK, now we actually call it dispersion relation. This term may not mean much to you now, but later in the discussion, you will find, aha! It really makes sense, and that we will talk about dispersion in the later lectures.

OK, so the conclusion from here is that, basically, if I have this distance that satisfies this translation symmetry, then what it tells us is that the normal modes, what looks like some kind of sinusoidal function, as we discussed last time. And also this-- OK, so this is actually the amplitude, what I am drawing here, this curve. And all those masses are only moving up and down, OK? As a function of time. And this is aj, and that is the oscillation frequency, which is actually the frequency of moving up and down, this kind of motion, is actually omega.

And also, we learned that omega is equal to omega of k. And that is actually decided by the length, and how distorted is this normal mode-- the shape of the normal mode? And this is actually determined by the k, which is actually the wave number, and of course you can also get the wavelength from two pi over k. OK? In short, if you give it a specific wave number or wavelength, than the oscillation frequency is already fixed because of the equation of motion, which we did, right, from the first diagram. Any questions?

OK. The last point which I would like to remind you is that, at this point, since we are talking about infinitely long systems, therefore all possible k are allowed. Right? Because, basically, you have an infinite number of coupled oscillators, and therefore you have an infinite number of normal modes. So all possible cases are allowed, and that actually because we have even an infinitely long system. After the break, which we will take a five minute break, we will discuss how to use infinitely long systems to actually understand a finite system.

So you will see that, actually I can use, now, this space translation symmetry, and to solve, in general, infinitely long systems. And I can actually even go back to find to a finite system and see what we can get from there. OK? So we will be back at 12:20. If you have any questions, I will be here

OK, welcome back, everybody. So we will continue the discussion. So there were a few questions asked during the break. So, the first question is related to how we actually arrive at this equation. And that is actually because-- OK, two t over ma is actually really happening in a diagonal term. Therefore, if you multiply m minus one k matrix and A matrix, which is actually shown there, then you will get this term, minus t over ma multiplied by aj minus one plus two t over ma times aj price minus t over ma times aj plus one. And that is actually why we can arrive at this expression. OK?

Then what happens afterward is that we found that aj can be factorized out, and they cancel.

And then now, my solution depends now upon the amplitude, and still omega is actually dependent on the k value, which we actually choose. OK. The second question is, why do I say k is the wave number? OK, where is that coming from? So that is because-- OK, so aj is proportional to exponential ijka. OK? It has a fancy name. If I take the real part, OK, as we did when we went to the description of physical systems, then you get cosine jka. OK?

And j times a is actually-- j is actually a label, right? Labeling which mass I am talking about. A is actually the distance between all those masses. j times a will give you the x location of the mass. So j times a is actually the the x position of the mass. OK? Therefore, if you accept that, this becomes cosine kx, and from there you will see immediately that k has a meaning, which is actually the wave number. OK. All right, is that? OK, so that was the questions raised, which I can quickly explain. So, what I am going to do now is that-- OK, we have solved, in general, an infinitely long system. What are actually the resulting normal modes of infinitely long systems?

It has an infinite number of normal modes, and we will wonder if I can actually borrow this infinitely long system and solve finite systems to see if I can arrive at the solution really quickly. OK? So the answer is actually yes. So if I consider a finite system that looks like this; so I have many, many little masses on this system and they are connected to each other by the center strings, which I prepared before, OK? And I call this the position in the y direction of this object y1, and then the next object y2, y3, etc. And I have an object in this system and both ends of the string are fixed on the wall. OK?

So I can actually now argue that the infinitely long system can help us with the understanding of this finite system. Why is that? That is because, now, I can assume that, huh, this is actually just part of an infinitely long system. All right. So I construct my infinitely long system, and now I nail the yth mass, I nail the y n plus one mass, and I fix that so that it cannot move, OK? So it's still an infinitely long system, but there are two interesting boundary conditions at j equal to zero and j equal to n plus one. OK. What are the two boundary conditions?

The first one is y zero equal to zero. OK? And the second condition is y n plus one equal to zero. OK? So there are two boundary conditions, so basically what I'm looking at is still an infinitely long system, but I require y zero and y n plus one to satisfy these two conditions. OK? And we will find that, huh, with this procedure, we can also solve this finite number of couple oscillators. The problem, in this case we, have coupled oscillators. OK? So the first thing which I would like to say is, based on the functional form, the functional form of omega square, now this is equal to four omega zero square sine square ka over two. OK?

What we actually have is that omega k is equal to omega minus k. OK? So both of them will give you the same angular frequency. OK, therefore, what does that mean? This means that linear combination of exponential ijka and the exponential minus ijka-- OK, linear combination of these two vectors-- is also an Eigenvector of m minus one k metrics. OK, so you can do linear combination of these two vectors. OK, so if we do that, now I can guess my solution will be like yj equal to real part of exponential i omega t plus phi.

I can now have a linear combination of exponential ijka and the exponential minus ijka. Basically, I have alpha exponential ijka plus beta exponential minus ijka. OK? And I would like to determine why that's actually alpha and beta which actually satisfy these boundary conditions, one and two. OK, so now I can use the first boundary condition, y zero equal to zero, right? So j equal to zero. Therefore, basically, what I get is, when j is equal to zero, then this is actually one and this is actually one, right? And this actually gives you y zero equal to zero. If y zero is equal to 0 at all times, no matter what t as you give it to this system, then basically you have alpha plus beta equal to zero, right? Because j is equal to zero. So you have alpha plus beta, and that has to be equal to zero.

Therefore, you can conclude that alpha is equal to minus beta. And I've reused the second boundary condition, y n plus one equal to zero, because I nailed this mass and then fixed that so that it cannot move. Then basically, what you get is y n plus one is equal to zero, then basically you have alpha exponential i n plus one ka plus-- okay, so beta is your equal to minus alpha, right? So basically, you can get minus exponential minus i n plus one ka, right? Multiplied by alpha. And now this is actually equal to zero.

Now we have the choice. We can actually set alpha to be equal to zero, but if I set alpha to be equal to zero, then beta is also zero. Then I have zero everywhere, right? Then there's no oscillation. And that's not fun, right? OK. Therefore, what I'm going to set is actually the second turn equal to zero. OK. The second turn, I can actually simplify that to be two i sine n plus one ka. And now this is actually equal to zero. OK. What is actually the condition of this thing equal to zero? Basically, n plus 1 is actually a given number, a is actually the distance between those masses, therefore, what I can actually change is the k value. Right?

So I can now solve this condition, and I will conclude that k will have to be equal to n times pi divided by N plus one. I hope you can see it. Where small n is equal to one, two, three, until capital N. So what does that mean? This means that-- OK, originally, before I introduced the boundary condition, this system is infinitely long, OK, and it has an infinite number of normal modes, right? But once I introduced these boundary conditions, which I actually require y zero equal to zero, because I fixed this point on the wall. Y m plus one equal to zero because I fixed, also, that point on the wall.

Something really happened. Now it actually gives us, first, the shape of the system when it is actually in the normal mode. Basically, the shape-- what I mean here is the amplitude as a function of j, OK? That's actually what I mean by shape, OK? The shape is now like a sine function. That's the first thing which we get from here. The second thing which we get here is that now, the k values are not arbitrary anymore. The k values are equal to n pi over n plus one. And the the small n is actually equal for one, two, three, until N. So now, once you actually fix this two point, you actually have only how many normal modes? N normal modes! Right?

So what I want to tell you is that, in general, the sinusoidal shape is actually fixed already by this translation symmetry argument. OK. And once we nail both sides-- actually, we also restrict ourselves to the discussion of only a few k which actually satisfy the boundary conditions. And if I plot all those normal modes as a function of i, basically what you can see from here, you can see if I have n equals to one and capital N equal to four in this case. OK, so I have four-- basically, I'm going to have four normal modes.

The first one will be like a really long wavelength one, when n is equal to one. And if I increase the small n value so that the k becomes bigger, then you can see that there is more distortion when this system is in one of the normal modes. And this shape is actually going to be oscillating up and down, instead of-- OK, so all those points are only moving up and down, right? Just a reminder. OK? And why do we have all those cases? Because of the boundary conditions. OK, any questions?

OK, so I have several other cases, which is open end and the closed end, and also the driven and the coupled oscillator examples. Also in the lecture notes, but unfortunately, we are will not be able to go over them, but I think they are very, very detailed, the notes in the lecture notes. OK. So, let me-- before I move on to the discussion of continuous systems, OK, I would like to discuss with you what we have learned so far. So what we have learned is that, if I have a symmetry which is a translation symmetry, and plus, we only limit ourselves in the discussion of oscillation. OK, in other words, we limit the amplitude so that it doesn't explode at the edge of the universe. OK.

And I will give you a beta value which is the functional form of exponential ika, and equation of motion can be derived from the first diagram. Once we entered the equation of motion, we can get m minus one k matrix, then we can derive omega square from this expression. And finally, we actually can simplify everything and then get the dispersion relation omega equal to omega k, which is a function of k, the wave number. Before we actually introduce boundary conditions to go from an infinitely long system to a finite system, all the k values are allowed. Once you introduce boundary conditions, you find that you only have a limited number of normal modes.

Second, the k value not continuous at any value any more, it becomes discrete. And only n values allowed from this exercise. And finally, what is actually the most general solution is actually the linear combination of all those normal modes, which we show here. And what is actually the ratio between all those normal modes? All those free coefficients are determined by initial conditions if you are given. OK? So that's actually what we have learned so far. And now I would like to make a leap of faith to see what happens. OK, what we are going to do is to introduce you to a continuous infinite number of coupled oscillators. OK?

So what does that mean? What I am going to do is to go from this-- so I have t and then a t, a lot of string and mass system, et cetera, et cetera, but I would like to go from there to just an infinitely long string with string tension t and some kind of density or mass. OK? We like to make it continuous to see what will happen. OK? So, just a reminder of the jth term of the m minus one k matrix operation. Basically, we have m minus one k times a, the j's term of m minus one k a. That is actually given by omega square aj. This is equal to t over ma minus aj minus one plus two aj minus aj plus one?

OK. So this is actually just a copy of that formula here. OK, so if I make it continuous-- OK, so that means what I'm going to get is omega square a but evaluated at position x, where x is actually equal to j times a. OK? And this will be equal to t over ma minus a, evaluated at x minus a plus two a x minus a x plus a. Now I am going to make this a very, very small, right? So that, when I make a very, very small, then it becomes a very, very continuous system. OK? So what I'm going to do is-- I can now make a go to zero.

I can now use Taylor's series fx plus delta x-- and just a reminder, if you do a Taylor expansion of this, you are going to get f of x plus delta x f prime x plus one over two factorial delta x square f double prime x. OK, so that means now, I can actually do a Taylor expansion of a x minus a and then a x plus a. So what I'm going to get is like this. So a x minus a will become a of x minus a a prime x plus one over two a square a double prime x. I can also do the same thing to do a Taylor expansion for a x plus a. a x plus a will be equal to a of x plus a prime x plus one over two a square a double prime x. OK.

Once I have done this, basically, then, I can calculate minus a x minus a plus two a of x minus a x plus a. OK, I'm just taking the middle term and copying it there. OK? If I use, now, this expression, OK, then basically I will see that, OK, a of x terms actually cancel, right? Because I have two a of x from these two terms, and they cancel with this two a x. OK? And also, a prime terms, also cancel, right? Because you have a x minus and a x plus a, therefore, the x prime turns cancel here. This is a minus sign, this is a plus sign. You see?

So therefore, what is actually left over is all those terms. This is going to give you a double prime x a squared, plus many other higher order terms, right? Because those are actually not completed, we have many, many higher order terms. OK? In the limit of x go to zero-- Yes.

STUDENT: Isn't the xa prime [INAUDIBLE]?

PROFESSOR YEN-JIE LEE: Yeah. Oh yeah, you are right. Thank you very much. OK, thank you for that. So there should be a minus sign in front of it, OK. So basically what we have is a double prime x times a square plus some higher order turn of a cubed, et cetera, et cetera. OK, since we are talking about the limit of a goes to zero, we can safely ignore all the higher order terms. OK? So now, if we go back to this equation, omega square a will become, basically, t over m minus t over ma a double prime x times a squared. OK, and then plus some higher order terms.

OK, now I can define rho l to be equal to m divided by a. OK. And I will I will have to be very careful when I go to the continuous limit, OK, I also do not want to make this system infinitely massive, right? Therefore, I would like to fix the rho l when I go to the continuous limit. So basically what I am doing is that I cut this system in half so that a becomes smaller and the mass also because smaller, so that rho l stays as a constant, OK, when I go to a continuous limit. OK, so what I'm going to get is omega square a will be equal to minus t over rho l a double prime x. And to write it explicitly, this is actually equal to minus t over rho l. I'll just square a partial x squared. OK? Because each prime is actually the differentiation which is spread to x, right?

OK. Don't forget-- what is actually this? This is actually m minus one ka, right? Equal to omega square a. And that this is actually just partial square a partial t square. Right? Because this is actually m minus one k matrix. It's actually originally-- going back to the original equation, we are actually solving m x double dot equal to minus kx problem, right? So there should be a minus sign there as well. Right? So mx double dot equal to minus kx, therefore x double dot will be equal to minus m minus one kx. Right?

So minus m minus one k matrix will be equal to x double dot, right? Therefore, what this is actually x double dot? It's basically-- in the current presentation, it's actually just partial square a, partial t square. OK? Can everybody accept this? OK, so now we have some sensibility going from a discrete system, which you have a length scale of a, to a continuous system, because a goes to zero. By a certain time, I fix the ratio of m and a so that the system doesn't grow too infinitely massive. OK, so if I do this, then in short, you get this equation. Partial square a partial t square, and that is equal to t over rho l partial square a partial x square. OK?

I can now define vp as equal to square root of t over rho l. OK? And this will become the p square partial square a partial x square. What is this? This is what equation? Wave equation! OK, you see that now? After all the hard work, OK, going from infinity long systems, discrete systems, then go to a continuous limit, we discovered the wave equation. This is probably the most important equation you actually learn, until now. More important than f equal to ma, right? Because you can listen to my lecture, even, if this equation didn't exist, right? Then I can not propagate a sound wave to your ear. Right?

And you cannot even see the black board, because the electromagnetic wave, which I will show in a later lecture, also kind of satisfies the same function or form. So I think this is an achievement and a highlight of today's class. We actually realize now, what we have been doing is really solving something related to the wave equation. And next time, what I would like to actually discuss with you is the solution of this wave equation. So here, I have-- again, last time you have seen this. This is a coupled oscillator system. It has 72 components. As a physicist, that's actually equal to an infinite number of coupled oscillators, OK. That's good enough.

And you can see that, if I do this, it does something really strange, right? You see a progressing wave going back and forth, and it disappears because of friction. OK? If I can construct something closer to a perfect coupled system without friction, what is going to happen is that this wave is going to be there bouncing back and forth forever. And that actually can be understood by the wave equation. And also, if I oscillate this system at a fixed frequency, you will see that these become standing waves.

And of course I can I do crazy things, I can oscillate and stop it and it becomes really, really complicated motion. And of course you are welcome to come here and play after the class. And next time, on Thursday, we are going to talk about the solution to this equation and how to understand all kinds of fancy motion this system can do, given by the nature. Thank you very much.