Reminder:

Maxwell's Equation in vacuum:

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = 0 \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} = \dfrac{1}{c^2} \dfrac{\partial \vec{E}}{\partial t} \end{cases}$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Resulting Wave Equations:

$$\begin{cases} \vec{\nabla}^2 \vec{E} = \dfrac{1}{c^2} \dfrac{\partial^2 \vec{E}}{\partial t^2} \\ \vec{\nabla}^2 \vec{B} = \dfrac{1}{c^2} \dfrac{\partial^2 \vec{B}}{\partial t^2} \end{cases}$$

We discussed plane harmonic wave solution

And you will show that the in general a progressing wave solution

$$\vec{E} = E_0 \hat{y} f(z - vt) \quad + \quad \text{corresponding } \vec{B} \text{ field}$$

also satisfies Maxwell's equations.

How do we transmit "information" ?

Simple harmonic wave : not useful.

We must use "pulses", chunks of localized energy in time.

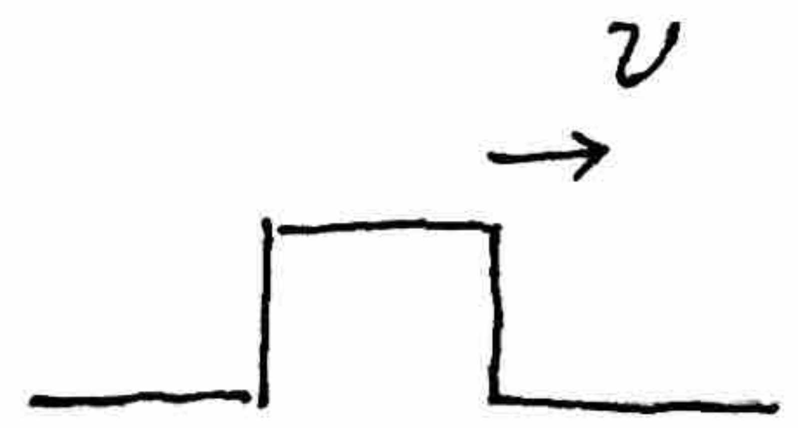
For instance :

We have learned :

$f(x - vt)$
$f(kx - \omega t)$ is a traveling wave moving in $+x$ direction and its shape is kept unchanged if and only if we are working on Non-dispersive medium.

$$\frac{\omega}{k} = v$$

Consider an ideal string :



$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial t^2}$$

where $\frac{\omega}{k} = v = \sqrt{\frac{T}{\rho_L}}$

If we create a square pulse , the square pulse will move at a constant speed v .

The shape of the square pulse doesn't change !

We call this string a non-dispersive medium

and the "dispersion relation" is

$$\omega = v \cdot k$$

(String tension is responsible for the restoring force)

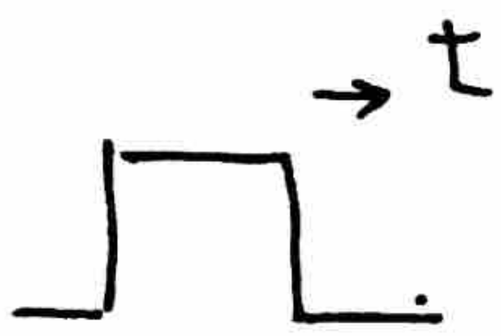
However , if we consider the "stifness" of the string ,

(for example , piano string)

If we bend a piano string , even when there is no tension , the string wants to restore to its original shape

Consider an ideal string: $\frac{\omega}{k} = v_p = \sqrt{\frac{T}{\rho_L}}$

$$\frac{\partial^2 \psi}{\partial t^2} = v_p^2 \frac{\partial^2 \psi}{\partial x^2}$$



shape doesn't change!

$\psi(x,t)$

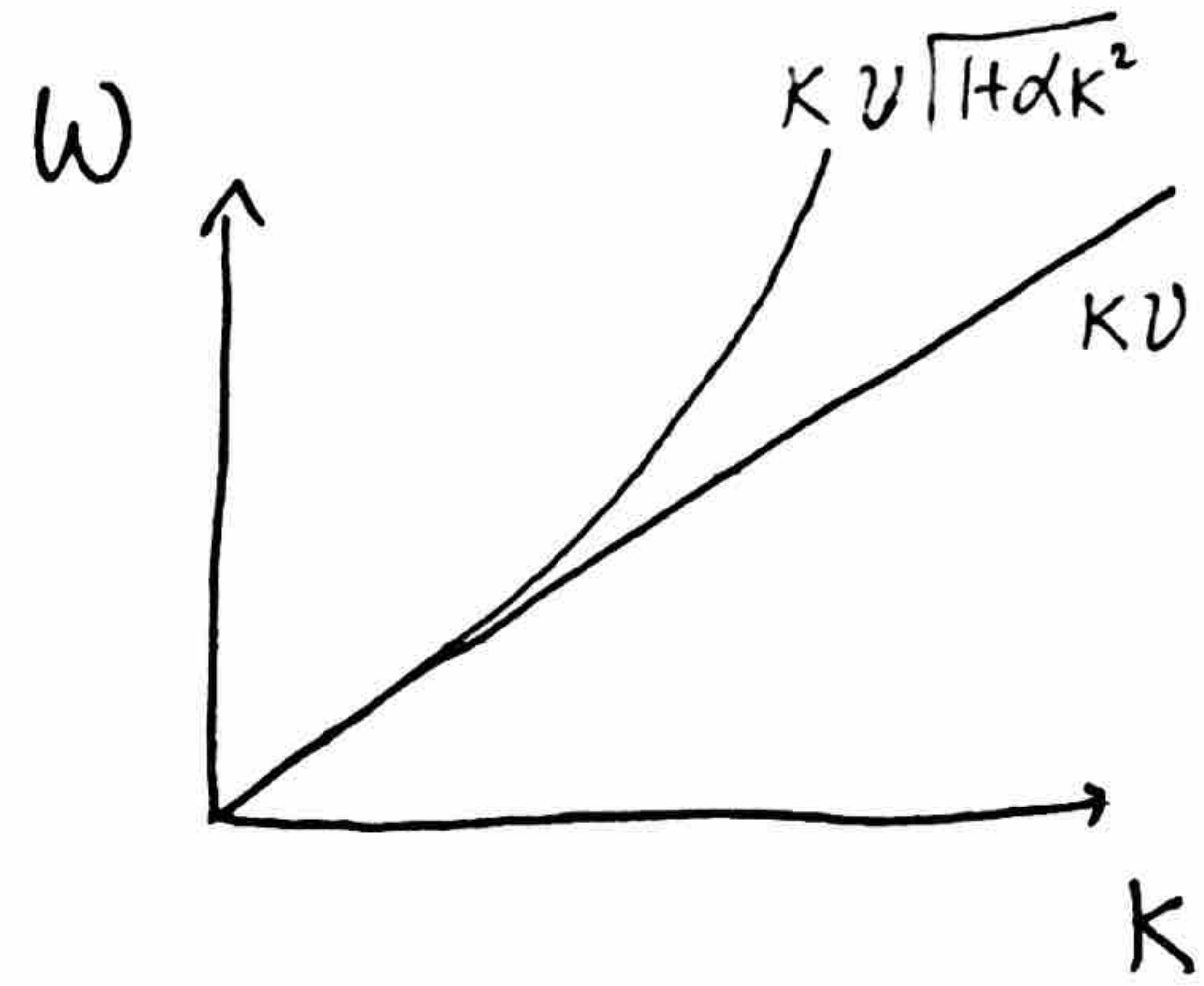
However, if we consider the "stifness" of the string, make it more realistic

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \left[\frac{\partial^2 \psi}{\partial x^2} - \underbrace{\alpha \frac{\partial^4 \psi}{\partial x^4}}_{\text{model.}} \right]$$ Note

Dispersion relation becomes: (Test function $A\cos(kx-\omega t)$)

$$\omega^2 = v^2 (k^2 + \alpha k^4)$$

$$\Rightarrow \frac{\omega}{k} = v\sqrt{1+\alpha k^2}$$ Not a constant v.s k !!



$k = 2\pi/\lambda$

Large k ⇒ short λ ⇒ a lot of distortion

⇒ higher speed V

Consequence:

Components with different k will be moving at different speed $v_p = \frac{\omega(k)}{k}$ if we consider

$$\sin(kx - \omega(k)t)$$

$\Rightarrow$ Dispersion !

(Demo)

Dispersion Mode / Alpha

1 0.02

Dispersion a variation of wave speed with wave length

Example : Addition of two progressing waves

$$\psi_1(x,t) = A \sin (k_1 x - \omega_1 t) \qquad v_1 = \frac{\omega_1}{k_1}$$

$$\psi_2(x,t) = A \sin (k_2 x - \omega_2 t) \qquad v_2 = \frac{\omega_2}{k_2}$$

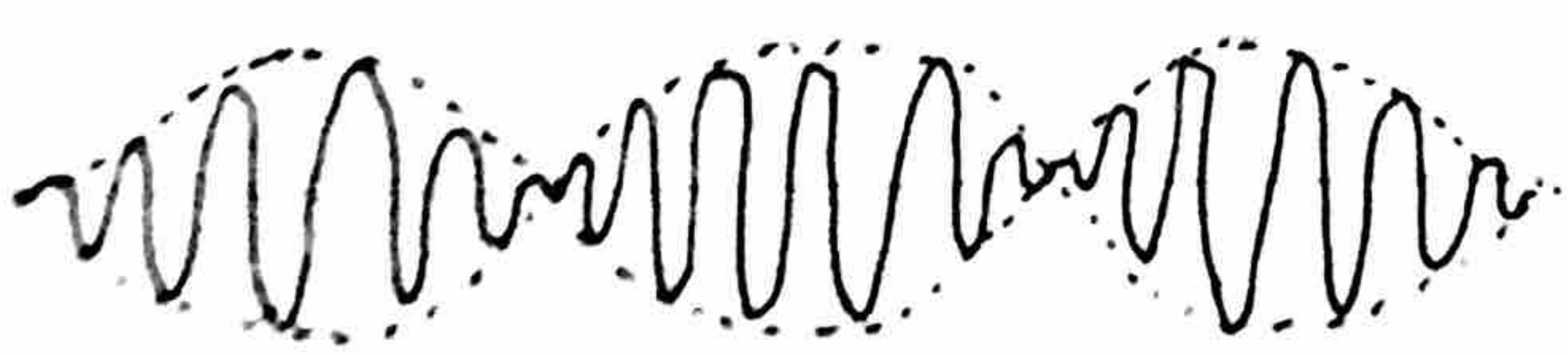
Adding ψ_1 and ψ_2 $\qquad \sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$

$$\Rightarrow \psi = \psi_1 + \psi_2$$

$$= 2A \sin \left(\frac{k_1+k_2}{2} x - \frac{\omega_1+\omega_2}{2} t \right) \cos \left(\frac{k_1-k_2}{2} x - \frac{\omega_1-\omega_2}{2} t \right)$$

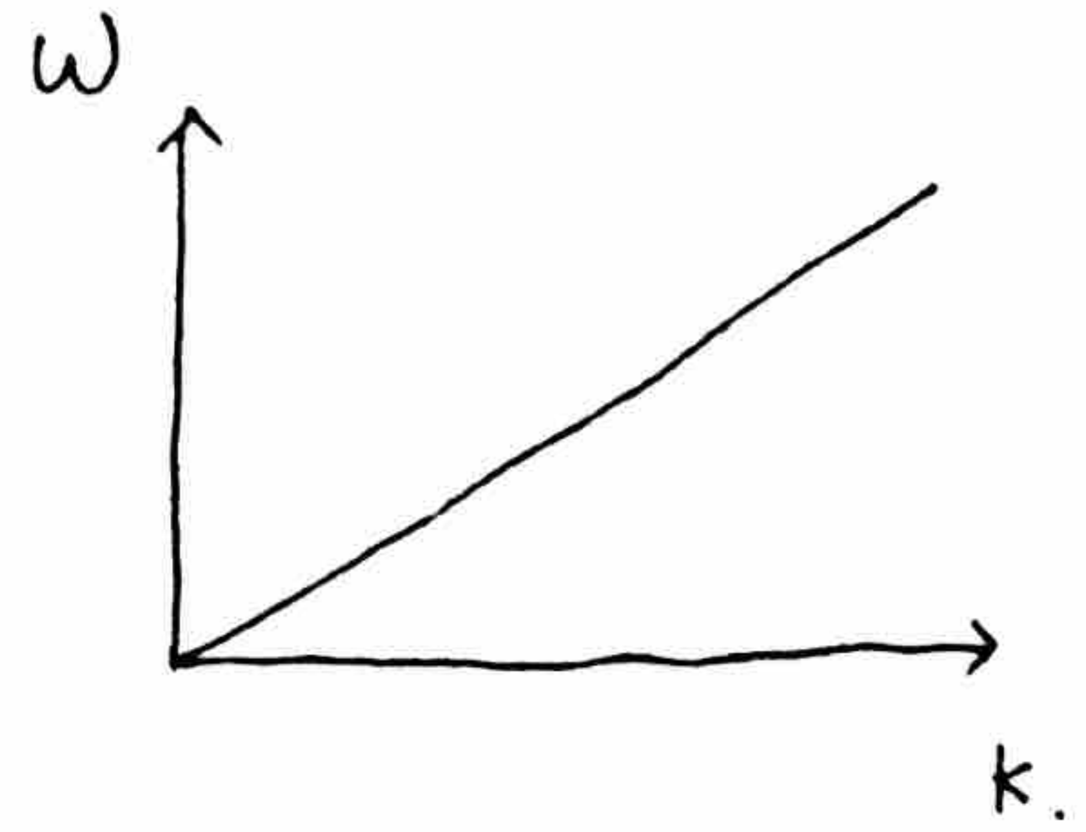
Assuming $k_1 \approx k_2 \approx k \qquad \omega_1 \approx \omega_2 \approx \omega$

Amplitude Modulation

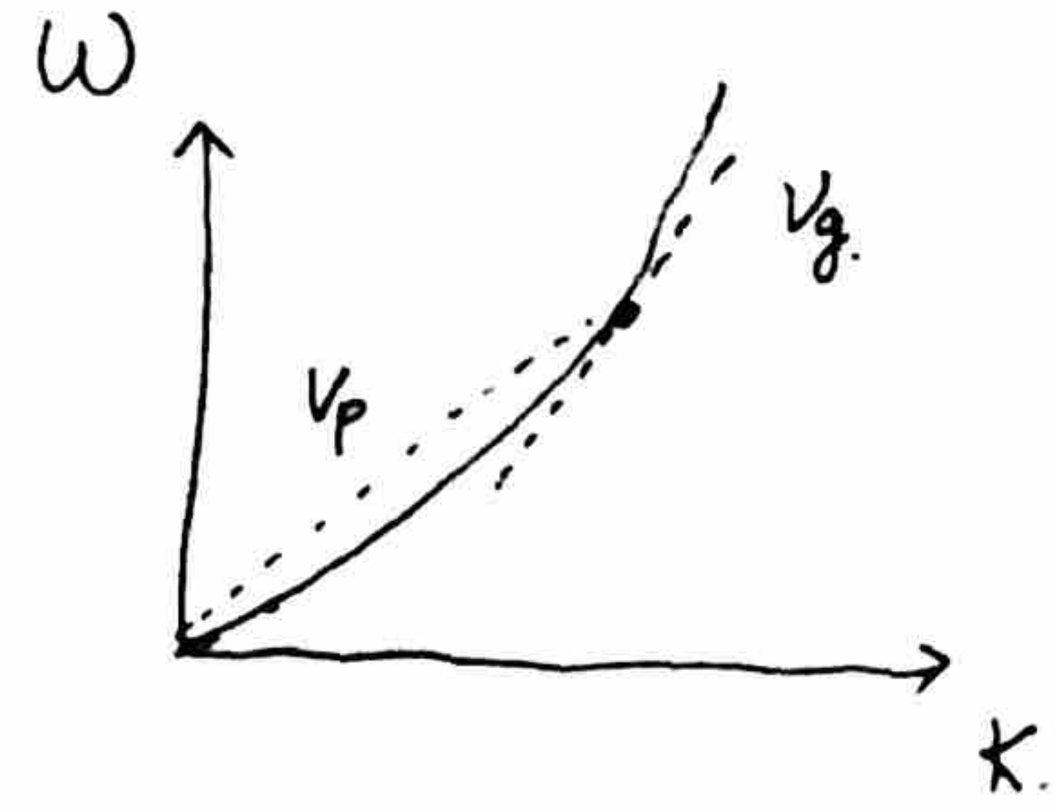


Phase velocity : $v_p = \frac{\omega}{k}$

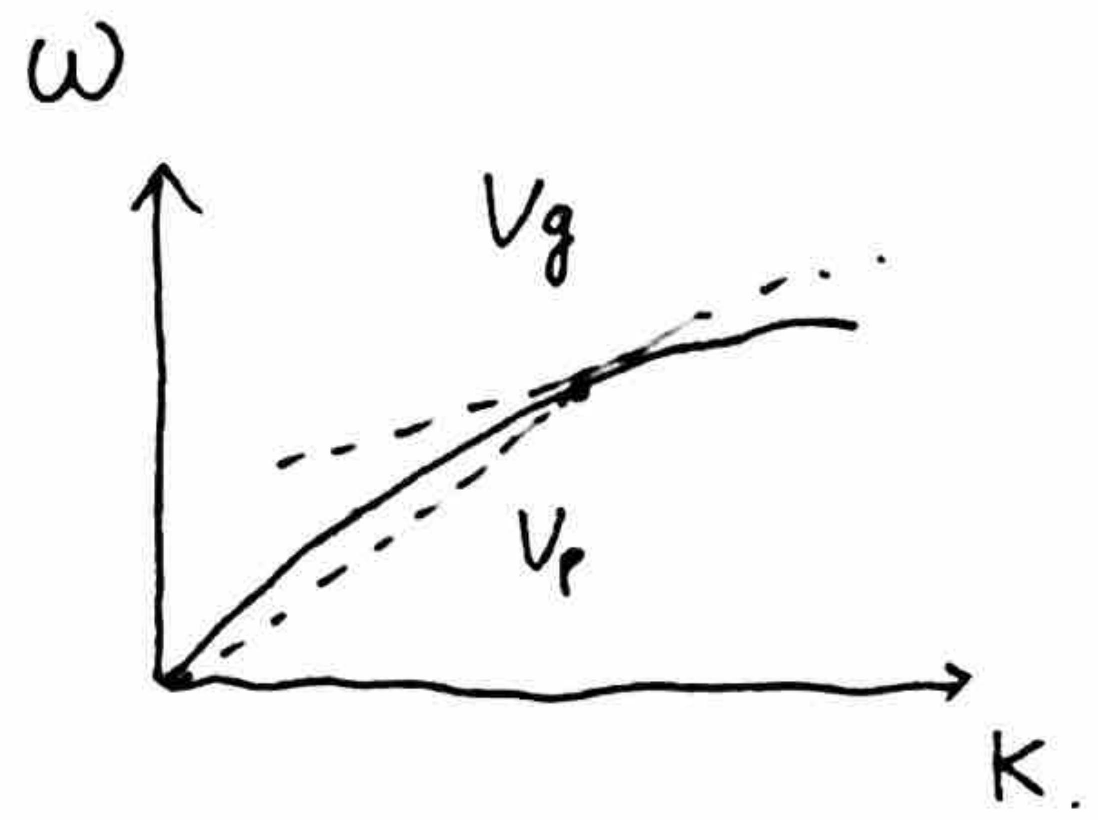
Group velocity $\quad v_g = \frac{(\omega_1 - \omega_2)}{(k_1 - k_2)} \approx \frac{d\omega}{dk}$



Non-dispersive medium



$V_g > V_p$

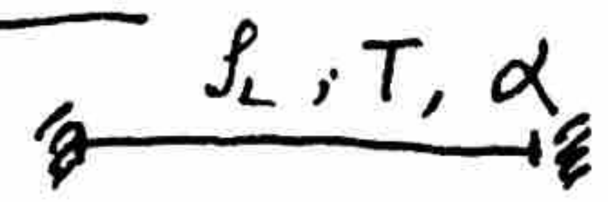


$V_p > V_g$

It is also possible that V_g goes to negative !

DENO
Group velocity

Bounded system :

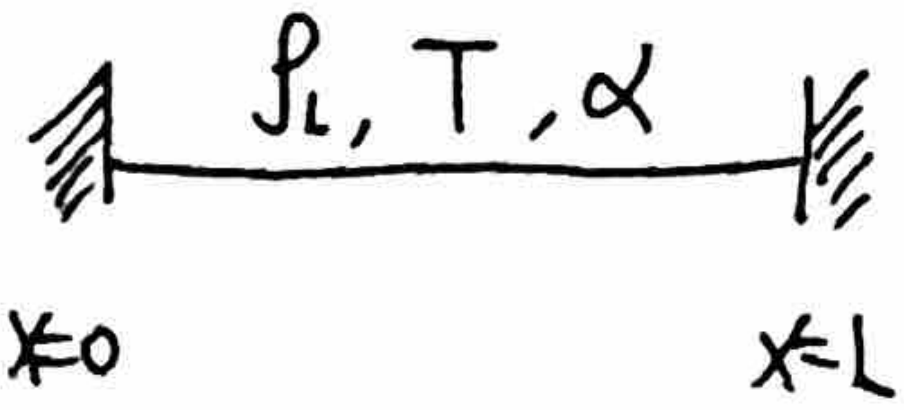


$$\psi(x,t) = \sum_m A_m \sin(K_m x + \alpha_m) \sin(\omega_m t + \beta_m)$$

$$\omega_m = \omega(K_m)$$

Then evolve as a function of time !

Now consider the boundary condition of this system:



$\psi(0,t)=0$ & $\psi(L,t)=0$

Similar to what we have solved before:

$$k_m = \frac{m\pi}{L} \quad , \quad \alpha_m = 0$$

Identical to the ideal string case ($\alpha=0$)

We learned that:

① The boundary condition "set" the k_m!

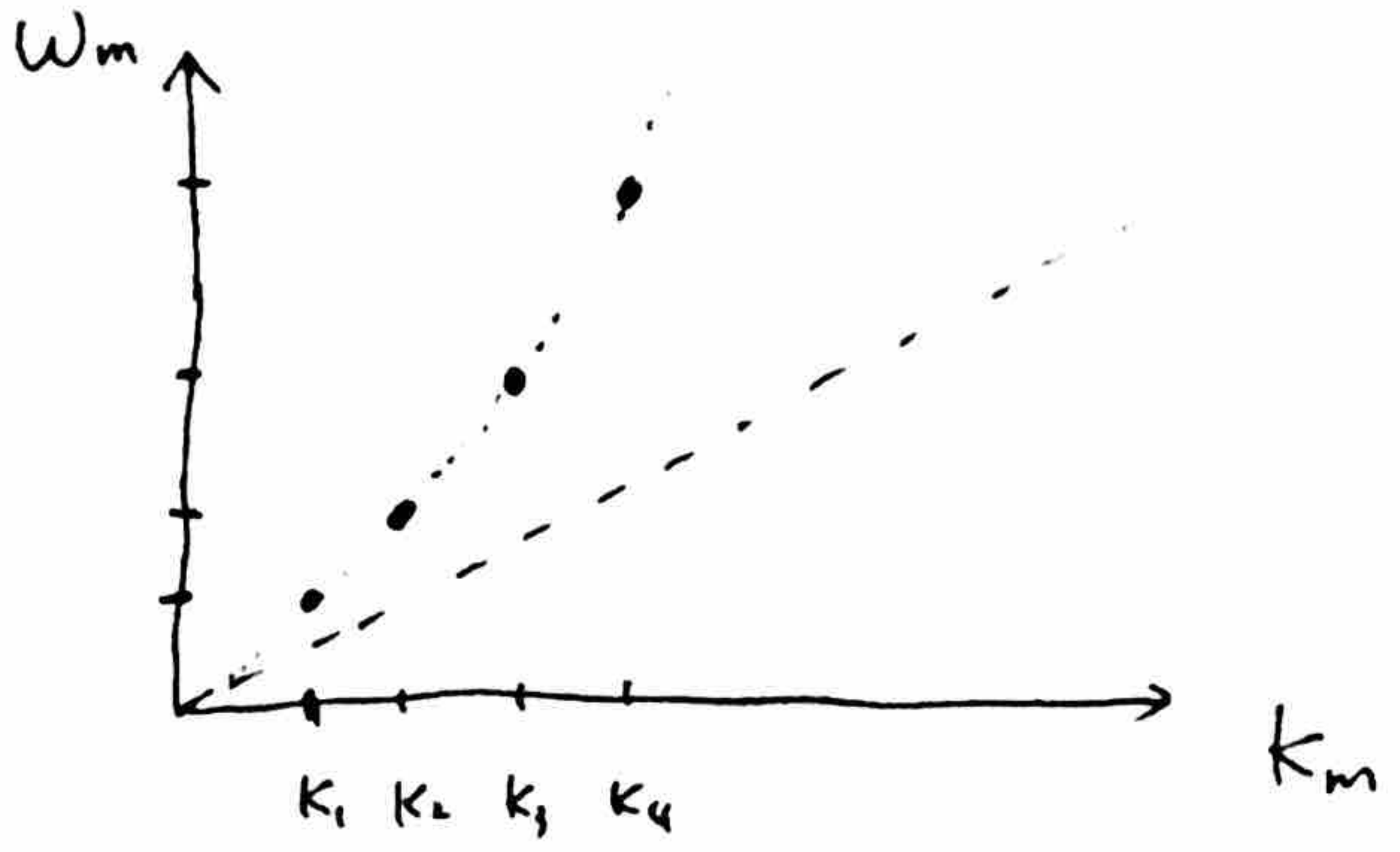
Does not depend on the dispersion relation $\omega(k)$

② The individual normal modes are oscillating

at $\omega_m = \omega(k_m)$ calculated by the dispersion relation:

Does depend on the dispersion relation!

If we plot the dispersion relation :

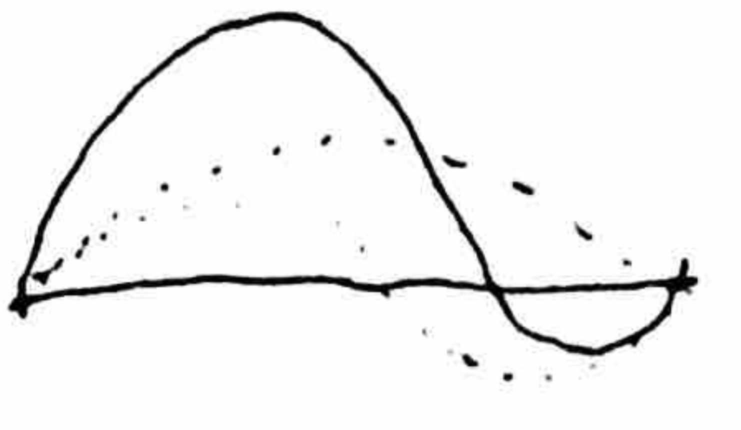


equally spaced ! But ω_m is not equally spaced.

Full solution

$$\psi(x,t) = \sum_m A_m \sin(k_m x + \alpha_m) \sin(\omega_m t + \beta_m)$$
$$= \sum_m \psi_m$$

Ex: $\psi(x,t) = \psi_1 + \psi_2$



In non-dispersive medium : the system go back to the original shape after $\frac{2\pi}{\omega_1}$

In dispersive medium : $\omega_2 \neq \omega_1$

Need to wait longer until the $T =$ least common multiple of $\frac{2\pi}{\omega_1}$ and $\frac{2\pi}{\omega_2}$