Massachusetts Institute of Technology
Physics 8.03 Fall 2016
Exam 1

Instructions

Please write your solutions in the white booklets. We will not grade anything written on the exam copy. This exam is closed book. No electronic equipment is allowed. All phones, blackberry, blueberry, raspberry Pi, tablets, computers etc. must be switched off.

Springs and masses:

$$m\frac{d^2}{dt^2}x(t) + b\frac{d}{dt}x(t) + kx(t) = F(t)$$

More general differential equation with harmonic driving force:

$$\frac{d^2}{dt^2}x(t) + \Gamma\frac{d}{dt}x(t) + \omega_0^2 x(t) = \frac{F_0}{m}\cos(\omega_d t)$$

Steady state solutions:

$$x_s(t) = A\cos(\omega_d t - \delta)$$

where

$$A = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \omega_d^2\Gamma^2}}$$

and

$$\tan\delta = \frac{\Gamma\omega_d}{\omega_0^2 - \omega_d^2}$$

General solutions:
For $\Gamma = 0$ (undamped system):

$$x(t) = R\cos(\omega_0 t + \theta) + x_s(t)$$

where R and θ are unknown coefficients.
For $\Gamma < 2\omega_0$ (under damped system):

$$x(t) = Re^{-\frac{\Gamma}{2}t}\cos\left(\sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}\,t + \theta\right) + x_s(t)$$

where R and θ are unknown coefficients.
For $\Gamma = 2\omega_0$ (critically damped system):

$$x(t) = (R_1 + R_2 t)e^{-\frac{\Gamma}{2}t} + x_s(t)$$

where R_1 and R_2 are unknown coefficients.
For $\Gamma > 2\omega_0$ (over damped system):

$$x(t) = R_1 e^{-\left(\frac{\Gamma}{2}+\sqrt{\frac{\Gamma^2}{4}-\omega_0^2}\right)t} + R_2 e^{-\left(\frac{\Gamma}{2}-\sqrt{\frac{\Gamma^2}{4}-\omega_0^2}\right)t} + x_s(t)$$

where R_1 and R_2 are unknown coefficients.
Coupled oscillators

$$F_j = -\sum_{k=1}^{n} K_{jk} x_k$$

Examples for $n = 2$

$$\mathcal{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\mathcal{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$\mathcal{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

Matrix equation of motion, matrices $\mathcal{M}$, $\mathcal{K}$, $\mathcal{I}$ are $n \times n$, vectors $\mathcal{X}$, $\mathcal{Z}$ are $n \times 1$.

$$\frac{d^2}{dt^2}\mathcal{X}(t) = -\mathcal{M}^{-1}\mathcal{K}\mathcal{X}(t)$$

$$\mathcal{Z}(t) = \mathcal{A}e^{-i\omega t}$$

$$(\mathcal{M}^{-1}\mathcal{K} - \omega^2\mathcal{I})\mathcal{A} = 0$$

To obtain the frequencies of normal modes solve:

$$det(\mathcal{M}^{-1}\mathcal{K} - \omega^2\mathcal{I}) = 0$$

For $n = 2$

$$det \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = M_{11}M_{22} - M_{12}M_{21}$$

If the system is driven by force one can find the response amplitudes $\mathcal{C}(\omega_d)$

$$\mathcal{F}(t) = \mathcal{F}_0 e^{-i\omega_d t}$$

$$\mathcal{W}(t) = \mathcal{C}(\omega_d)e^{-i\omega_d t}$$

$$\mathcal{C}(\omega_d) = \begin{bmatrix} c_1(\omega_d) \\ c_2(\omega_d) \end{bmatrix}$$

$$(\mathcal{M}^{-1}\mathcal{K} - \omega_d^2\mathcal{I})\mathcal{C}(\omega_d) = \mathcal{F}_0$$

solving the equation above one can find the response amplitudies for the first $(c_1(\omega_d))$ and second $(c_2(\omega_d))$ objects in the system.

Reflection symmetry matrix:

$$\mathcal{S} = \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix}$$

Eigenvalues (β) and eigenvectors $(\mathcal{A})$ of this 2×2 $\mathcal{S}$ matrix:

(1) $\beta = -1$, $\mathcal{A} = \begin{matrix} 1 \\ 1 \end{matrix}$

(2) $\beta = 1$, $\mathcal{A} = \begin{matrix} 1 \\ -1 \end{matrix}$

1D infinite coupled system which satisfy space translation symmetry:

Given a eigenvalue β, the corresponding eigenvector is

$$A_j = \beta^j A_0$$

where

$$A_j(A_0)$$

is the normal amplitude of jth(0th) object in the system.

Consider an one dimentional system which consists infinite number of masses coupled by springs, β can be written as $\beta = e^{ika}$ where k is the wave number and a is the distance between the masses.

Kirchoff's Laws (be careful about the signs!)

$$\text{Node}: \sum_i I_i = 0 \ \text{Loop}: \sum_i \Delta V_i = 0$$

$$\text{Capacitors}: \Delta V = \frac{Q}{C} \quad \text{Inductors}: \Delta V = -L\frac{dI}{dt} \quad \text{Current}: I = \frac{dQ}{dt}$$

Problem 1 (30 pts)

Solve the following short questions.

A. A graduate student was performing an experiment on a damped pendulum which was constrained to move only in the x direction. The position of the pendulum as a function of time is shown in Figure 1. Based on the experimental data, is this an underdamped, an over-damped, or a critically damped oscillator? (5 pts)

B. Two vibrations along the same string are described by the equation $y_1(t) = A\cos(3\pi t)$ and $y_2(t) = A\cos(4\pi t)$ where t is in seconds and A is 0.01 cm. Find the beat period of the superposition of the two. (5 pts)

C. A heavy pendulum was oscillating under water with angular frequency ω_1 (underdamped). Will the oscillation frequency increase, decrease or stay the same if the whole system is taken out of the water? Explain why. (5 pts)

D. Consider an oscillator with three masses arranged in a line and connected with springs. The oscillator has a mirror symmetry $x_1 \to -x_3$, $x_3 \to -x_1$ and $x_2 \to -x_2$. Write the symmetry matrix $\mathcal{S}$ for this system. (5 pts)

E. Explain briefly why a driven under-damped oscillator often exhibits large, apparently erratic, responses when the driving mechanism is first turned on? (5 pts)

F. Consider a simple mechanical oscillator of mass m attached to a spring of constant k. One may draw a close analogy between this system and a single loop circuit with inductor L and capacitor C connected in series. Both oscillators follow mathematically equivalent equations. What is the electrical equivalent of the mass m in the spring-mass system and of the spring constant k? (5 pts)

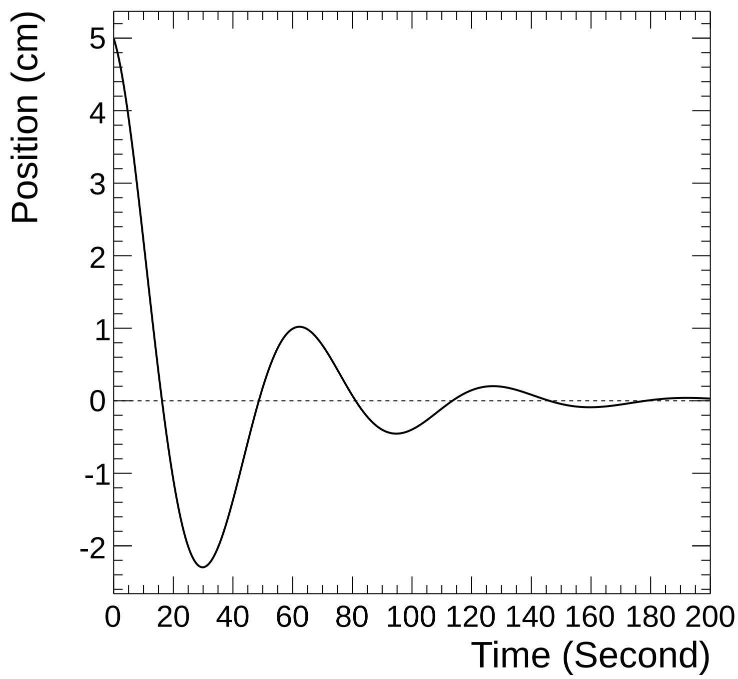

Figure 1: Displacement of a pendulum with respect to the equilibrium position

Problem 2 (30 pts)

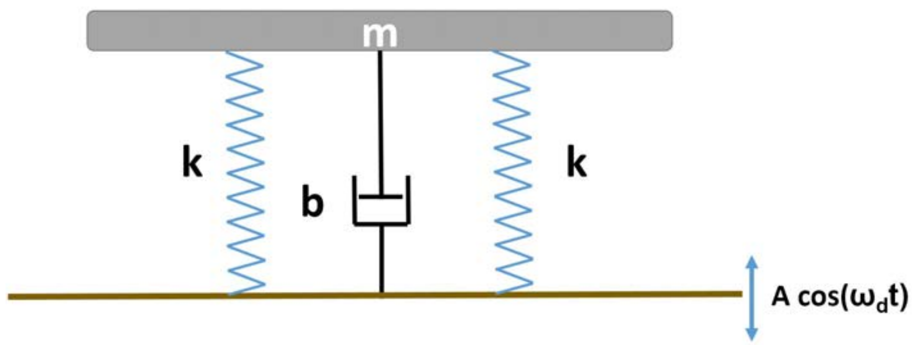

Figure 2: Vibration free table

Many precision scientific measurements require vibration free tables. One example is shown in Figure 2. Consider a table of mass m supported by 2 ideal springs with spring constant $k = \frac{1}{2}m\omega_0^2$ and damped by a damper with damping force $F = -m\Gamma v$ where v is the relative velocity between the floor and the table (NOT the velocity of the table). Assume that the system is underdamped. Both the spring and the damper are firmly attached to the table and to the floor. An earthquake happened and the floor is vibrating harmonically along the vertical direction with frequency ω_d and amplitude A

$$y_g(t) = A\cos(\omega_d t)$$

a. Write the equation of motion of the table in terms of $y(t)$, assuming $y = 0$ represents the equilibrium position of the table before the earthquake. (10 pts)

b. Find the amplitude of steady state vibrations of the table as a function of floor vibration frequency ω_d. (10 pts)

c. Make a sketch of the steady state amplitude as a function of ω_d, indicate the amplitude value at $\omega_d \sim 0$ and $\omega_d \to \infty$. (10 pts)

Problem 3 (40 pts)

For the system of three masses shown in the Figure 3, the ideal springs and the pendulum rods are massless, and x_1 and x_2 and x_3 are measured from the position of static equilibrium (i.e., the spring is relaxed and the pendulum hangs vertically). The whole setup is prepared on Earth with gravitation force pointing downward. We also assume that the oscillation amplitudes of the three masses are very small and there is no damping or friction in the system. You can also safely assume that the masses are only moving in horizontal direction.

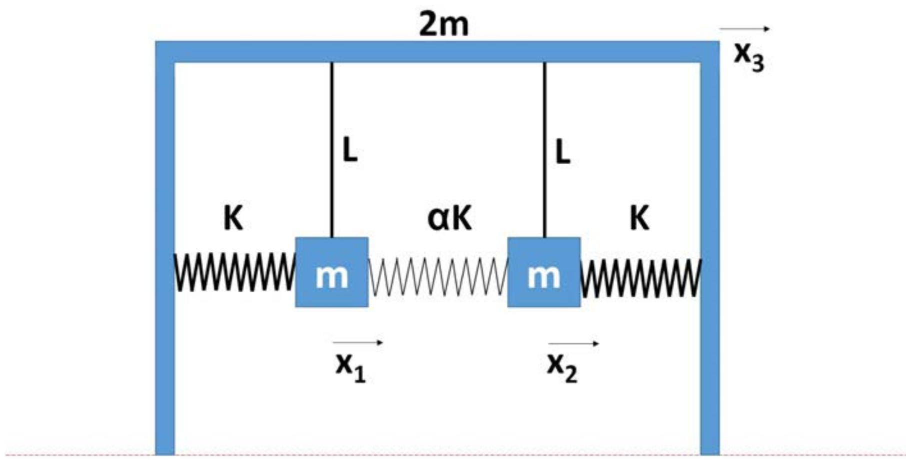

Figure 3: Coupled Pendulum and Ideal Spring Oscillators

a. In the beginning of the experiment, the third mass (the U shape one with mass $2m$) is fixed at $x_3 = 0$. Derive the coupled equations of motions for the positions of the two masses (x_1 and x_2). (6 pts)

b. Write down the 2×2 $\mathcal{M}^{-1}\mathcal{K}$ matrix (4 pts)

c. Solve the two normal mode frequencies. (6 pts)

d. Evaluate the amplitude ratios for each normal mode, describe the motion of each normal mode by a sketch. (8 pts)

e. Write down the general expression for motion of the two masses in terms of the normal mode frequencies and four unknown coefficients. (4 pts)

f. From now on, the third mass (with mass $2m$) is released and free to move in the x direction. Derive the coupled equations of motion for the positions of the three masses (x_1, x_2 and x_3) without solving it. (8 pts)

g. Describe the motion of the three normal modes by a sketch without solving the coupled equations of motion. (4 pts)