

## 8.03 Lecture 3

Summary: solution of  $\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = 0$

(0):  $\Gamma = 0$  No damping:

$$\theta(t) = A \cos(\omega_0 t + \alpha)$$

(1):  $\omega_0^2 > \frac{\Gamma^2}{4}$  Underdamped Oscillator:

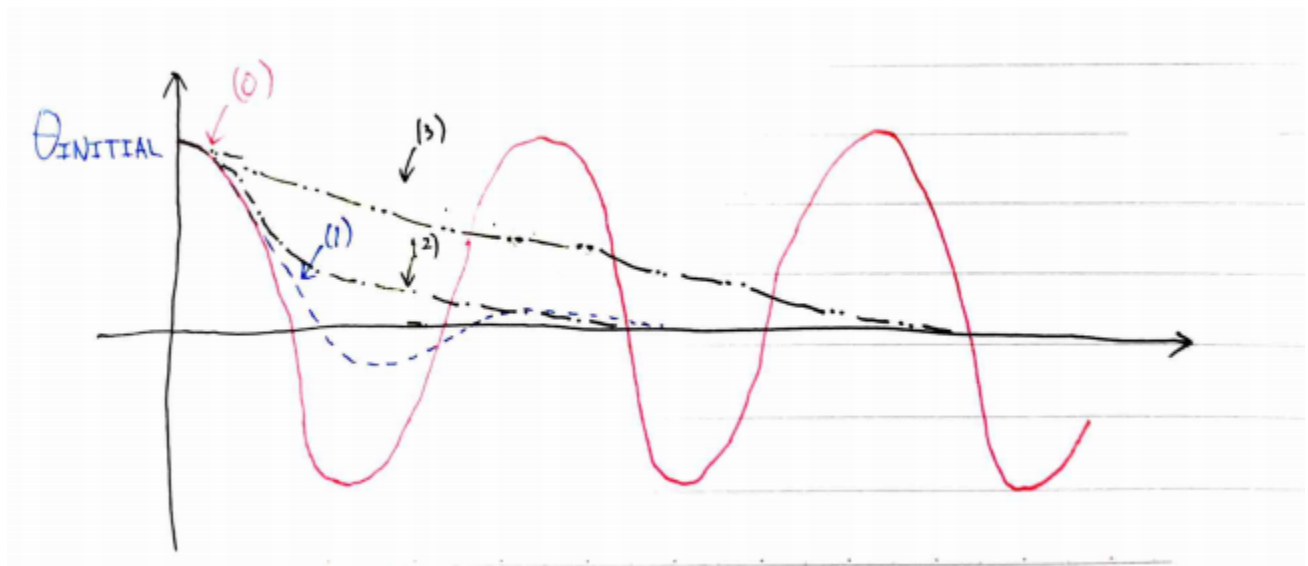
$$\theta(t) = Ae^{-\Gamma t/2} \cos(\omega t + \alpha) \text{ where } \omega = \sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}$$

(2):  $\omega_0^2 = \frac{\Gamma^2}{4}$  Critically damped oscillator:

$$\theta(t) = (A + Bt)e^{-\Gamma t/2}$$

(3):  $\omega_0^2 < \frac{\Gamma^2}{4}$  Overdamped Oscillator:

$$\theta(t) = Ae^{-(\Gamma/2+\beta)t} + Be^{-(\Gamma/2-\beta)t} \text{ where } \beta = \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}$$



Continue from lecture 2:

Now we are interested in giving a driving force to this rod:



Assume that the force produces a torque:

$$\tau_{DRIVE} = d_0 \cos \omega_d t$$

Total torque:

$$\tau(t) = \tau_g(t) + \tau_{DRAG}(t) + \tau_{DRIVE}(t)$$

Equation of motion:  $\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = \frac{d_0}{I} \cos \omega_d t$

Where, from last lecture, we have defined:

$$\Gamma \equiv \frac{3b}{ml^2} \quad \omega_0 \equiv \sqrt{\frac{3g}{2l}}$$

Where  $\Gamma$  is the size of the drag force and  $\omega_0$  is the natural angular frequency (i.e., without drive). Also, define  $f_0 \equiv \frac{d_0}{I}$ . Now our equation of motion reads:

$$\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = f_0 \cos \omega_d t$$

We would like to construct something to “cancel”  $\cos \omega_d t$ . Idea: use complex notation:

$$\ddot{z} + \Gamma\dot{z} + \omega_0^2 z = f_0 e^{i\omega_d t}$$

Guess:

$$z(t) = A e^{i(\omega_d t - \delta)}$$

where the  $\delta$  is designed to cancel  $e^{i\omega_d t}$ . It takes some time for the system to “feel” the driving torque. Taking our derivatives gives us:

$$\begin{aligned} \dot{z}(t) &= i\omega_d z \\ \ddot{z}(t) &= -\omega_d^2 z \end{aligned}$$

Insert these results into the equation of motion:

$$\begin{aligned} (-\omega_d^2 + i\omega_d\Gamma + \omega_0^2)z(t) &= f_0 e^{i\omega_d t} \\ (-\omega_d^2 + i\omega_d\Gamma + \omega_0^2)A e^{i(\omega_d t - \delta)} &= f_0 e^{i\omega_d t} \\ (-\omega_d^2 + i\omega_d\Gamma + \omega_0^2)A &= f_0 e^{i\delta} \\ &= f_0 (\cos \delta + i \sin \delta) \end{aligned}$$

Since this is a complex equation, we can solve for  $A$  and  $\delta$

Real part:  $(\omega_0^2 - \omega_d^2)A = f_0 \cos \delta$

Imaginary part:  $\omega_d\Gamma A = f_0 \sin \delta$

Squaring both of these equations and adding them together yields:

$$\begin{aligned} A^2 [(\omega_0^2 - \omega_d^2) + \omega_d^2\Gamma^2] &= f_0^2 \\ A(\omega_d) &= \frac{f_0}{\sqrt{(\omega_0^2 - \omega_d^2) + \omega_d^2\Gamma^2}} \end{aligned}$$

Dividing the imaginary part by the real part yields:

$$\begin{aligned} \tan \delta &= \frac{\Gamma\omega_d}{\omega_0^2 - \omega_d^2} \\ \Rightarrow \theta(t) = \text{Re}[z(t)] &= A(\omega_d) \cos(\omega_d t - \delta(\omega_d)) \end{aligned} \tag{1}$$

Where both  $A(\omega_d)$  and  $\delta(\omega_d)$  are functions of  $\omega_d$ .

No free parameter?! Actually, this is the a particular solution. The full solution (if we prepare the system in the “underdamped” mode) is:

$$\theta(t) = A(\omega_d) \cos(\omega_d t - \delta) + B e^{-\Gamma t/2} \cos(\omega t + \alpha)$$

Where the left side with amplitude  $A$  is the steady state solution and the right side with amplitude  $B$  will die out as  $t \rightarrow \infty$ .

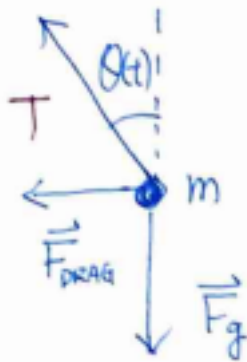
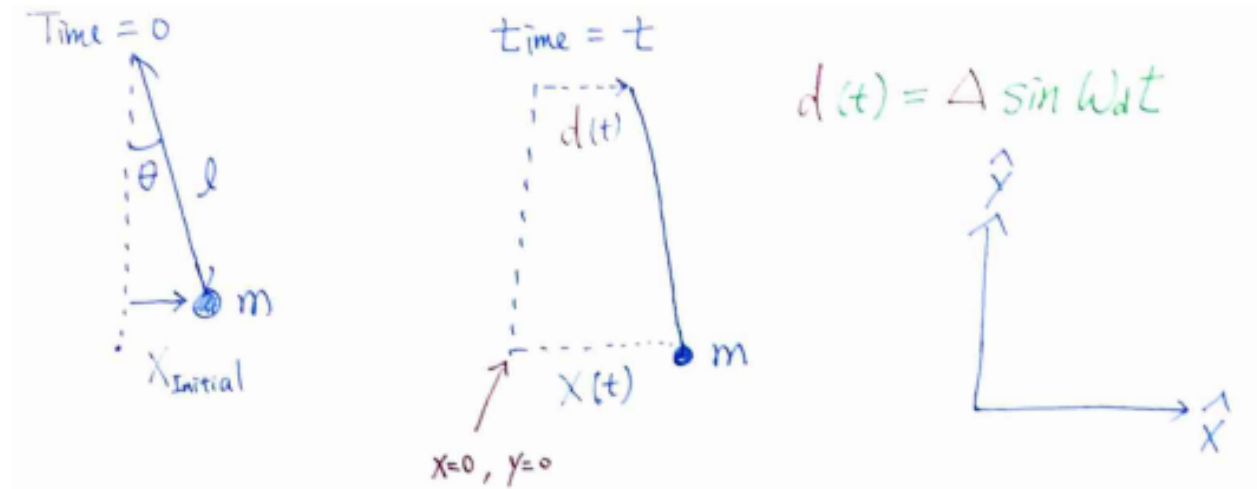
You may be confused with so many different  $\omega$ 's!! To clarify:

$\omega_0$  is the “natural angular frequency.” In our example with the rod,  $\omega_0 = \sqrt{3g/2l}$

$\omega$ : this frequency is lower if there is a drag force. It is defined by the equation  $\omega = \sqrt{\omega_0^2 - \Gamma^2/4}$

$\omega_d$  is the frequency of the driving torque or force

### Example: Driving a pendulum



← Force diagram

$$\vec{F}_{DRAG} = -b\dot{x}\hat{x}$$

$$\vec{F}_g = -mg\hat{y}$$

$$\vec{T} = -T \sin \theta \hat{x} + T \cos \theta \hat{y}$$

Take a small angle approximation:  $\sin \theta \approx \theta = \frac{x-d}{l}$  and  $\cos \theta \approx 1$

This implies:

$$\vec{T} \approx -T \frac{x-d}{l} \hat{x} + T \hat{y}$$

In the  $\hat{x}$  direction we have

$$m\ddot{x} = -b\dot{x} - T\frac{x-d}{l}$$

and in the  $\hat{y}$  direction we have

$$0 = m\ddot{y} = -mg + T$$

where the force has to be zero because there is no vertical motion (assuming a small angle). We now know  $mg = T$ .

Setting up our equation of motion we have

$$m\ddot{x} + b\dot{x} + \frac{mg}{l}x = \frac{mg}{l}\Delta \sin \omega_d t$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{g}{l}x = \frac{g\Delta}{l} \sin \omega_d t$$

To compare with our previous solution, define  $\Gamma \equiv b/m$ ,  $\omega_0^2 \equiv g/l$ , and  $f_0 \equiv g\Delta/l$  to give

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = f_0 \sin \omega_d t$$

Let us examine the amplitude:

$$A(\omega_d) = \frac{f_0}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \omega_d^2 \Gamma^2}}$$

There are a few cases we need to consider:

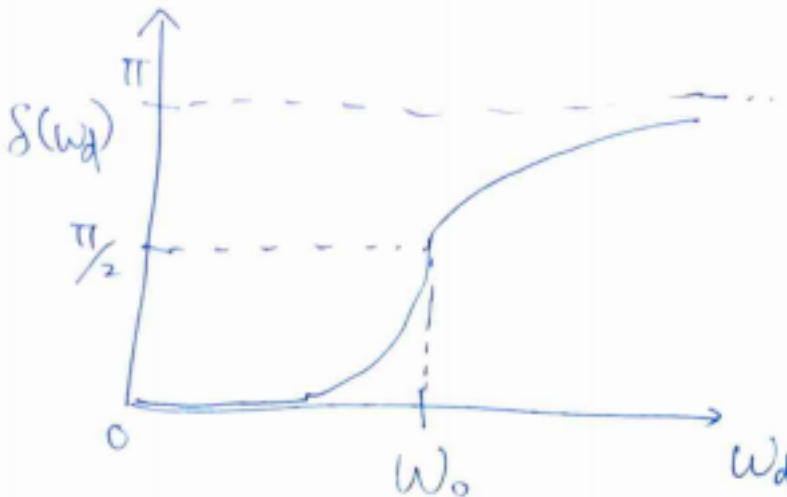
(1)  $\omega_d \rightarrow 0$

$$A(\omega_d) = \frac{f_0}{\omega_0^2} = \frac{g\Delta/l}{g/l} = \Delta$$

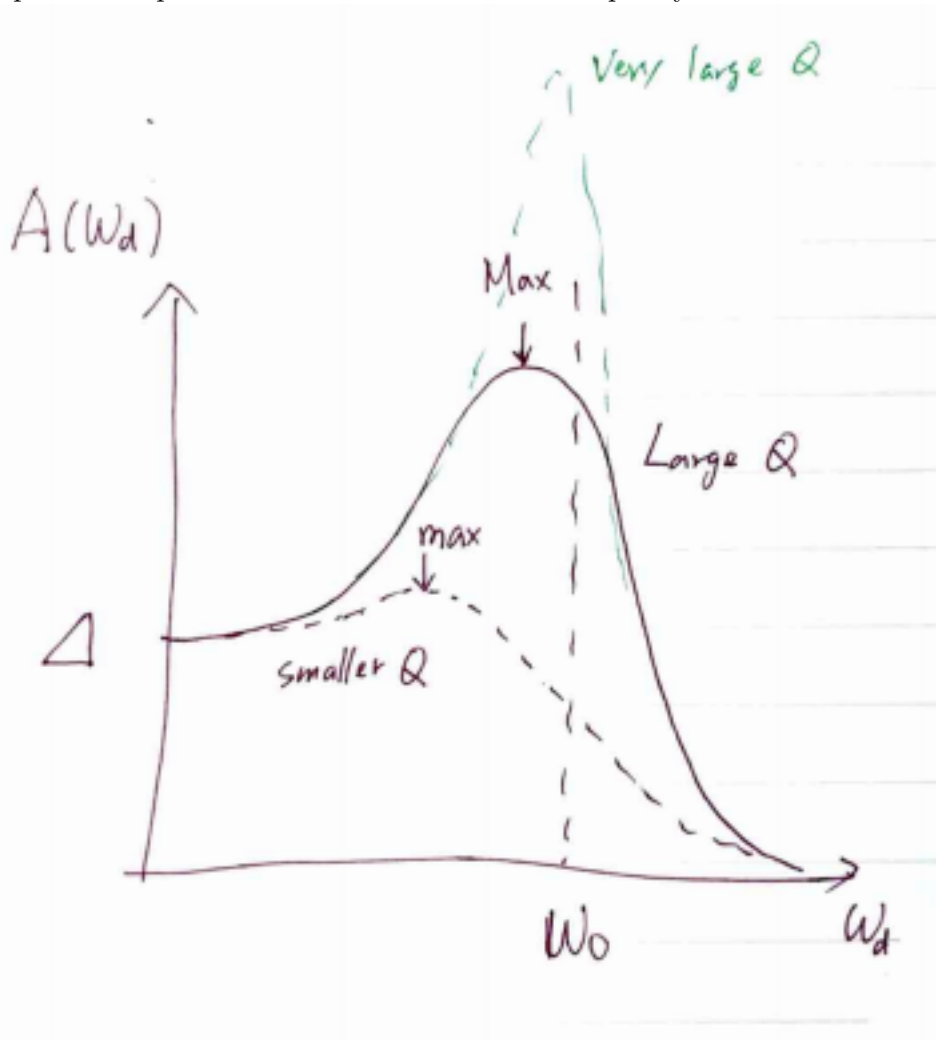
The amplitude will simply be the amplitude of the initial displacement. If the drive frequency is zero then  $\tan \delta = 0 \rightarrow \delta = 0$ .

(2)  $\omega_d \rightarrow \infty$

$A(\omega_d) \Rightarrow 0$  and  $\tan \delta \rightarrow \infty$  therefore  $\delta = \pi$



A plot of the phase as a function of the drive frequency.



A plot of the amplitude as a function of the drive frequency.

There is a third possibility:

**(3)**  $\omega_d \approx \omega_0$

This is called driving “on resonance.” Even a small  $\Delta$  can produce a large  $A$ , amplitude:

$$A(\omega_0) = \frac{f_0}{\omega_0 \Gamma} = \frac{\omega_0^2 \Delta}{\omega_0 \Gamma} = \frac{\omega_0}{\Gamma} \Delta = Q \Delta$$

Where  $Q \equiv \omega_0 / \Gamma$  and is a large parameter which gives a large amplitude.

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