8.03 Lecture 15

Review: Fourier Transform and Narrow band signal transmission.

$$f(t) = f_s(t)\cos\omega_0 t$$

In the small bandwidth approximation we have

$$\psi(x,t) \approx \mathrm{Re}\Big[\underbrace{f_s(t - x/v_g)}_{envelope}\overbrace{e^{-i(t-x/v_g)\omega_0}}^{carrier}\Big]$$

Where the envelope is traveling at group velocity $v_g = \left.\frac{d\omega}{dk}\right|_{\omega_0}$ and the carrier is traveling at phase velocity $v_p = \frac{\omega_0}{k_0}$

AM radio: typical $\omega_0 : 0.3 - 30MHz$ and $\Delta\omega : 5kHz$. We have $\Delta\omega \ll \omega_0$ Makes sense!

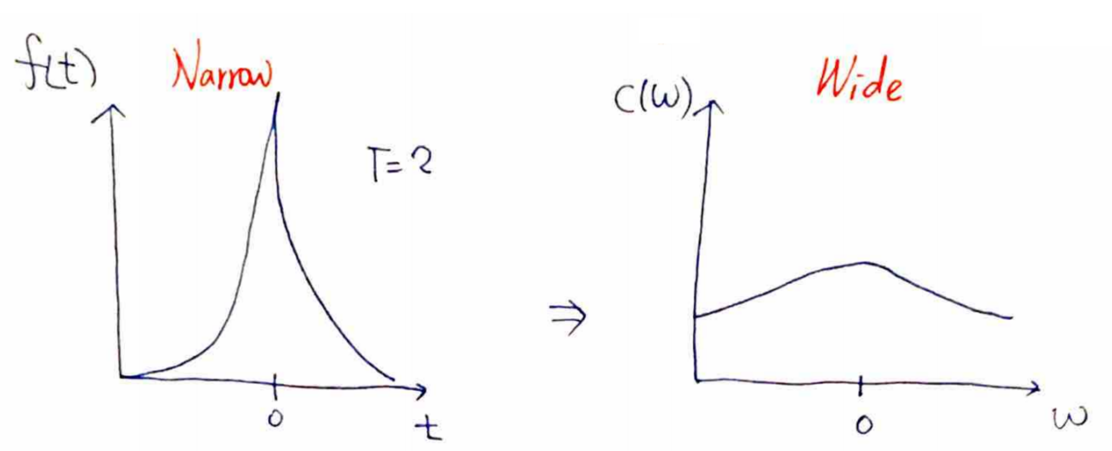
Example:

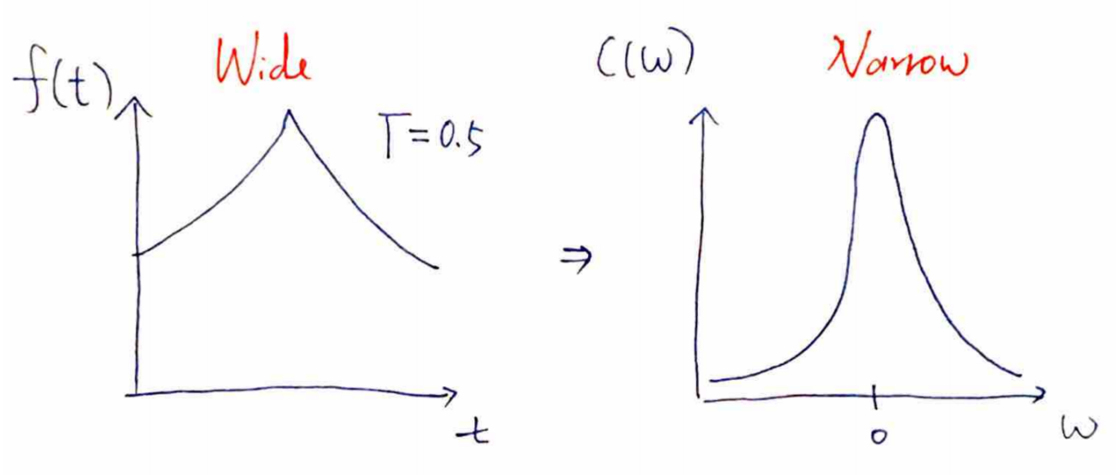
$$f(t) = e^{\Gamma|t|}$$

What will be the corresponding $C(\omega)$? From the previous lecture:

$$\begin{aligned}
C(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ e^{-\Gamma|t|}e^{i\omega t} \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{0} dt\ e^{+\Gamma t}e^{i\omega t} + \int_{0}^{\infty} dt\ e^{-\Gamma t}e^{i\omega t}\right] \\
&= \frac{1}{2\pi}\left[\frac{1}{\Gamma + i\omega} + \frac{i}{\Gamma - i\omega}\right] \\
&= \frac{1}{2\pi}\frac{2\Gamma}{\Gamma^2 + \omega^2} \\
&= \frac{\Gamma}{\pi(\Gamma^2 + \omega^2)}
\end{aligned}$$

If we plot $C(\omega)$ as a function of ω





Large Γ gives narrow pulse ($f(t)$ narrow) but one will get wide $C(\omega)$
Small *Gamma* gives wide pulse ($f(t)$ wide) but will get narrow $C(\omega)$
In your pset, you will work on another function form: the Gaussian wave. We can demonstrate this using a precie mathematical definition of the spread of the signal.

1. We define the intensity of the signal to be proportional to $|f(t)|^2$

2. Average value of any function weighted with the signal's intensity

$$\langle g(t)\rangle = \frac{\int_{-\infty}^{\infty} dt\ g(t)|f(t)|^2}{\int_{-\infty}^{\infty} dt\ |f(t)|^2}$$

3. Spread of time:

$$\Delta t^2 \equiv \langle [t - \langle t\rangle]^2\rangle = \frac{\int_{-\infty}^{\infty} dt\ \left|(t-\langle t\rangle)f(t)\right|^2}{\int_{-\infty}^{\infty} dt\ |f(t)|^2}$$

mean square deviation from the average time. Similarly spread of the frequency spectrum

$$\Delta\omega^2 \equiv \langle[\omega - \langle\omega\rangle]^2\rangle$$

We will prove

$$\Delta\omega \cdot \Delta t \geq 1/2$$

4. We also realize that

$$\begin{aligned}\int_{-\infty}^{\infty} d\omega\, \omega C(\omega)e^{-i\omega t} &= i\frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\omega\, C(\omega)e^{-i\omega t}\\ &= i\frac{\partial}{\partial t}f(t)\end{aligned}$$

$$\Rightarrow \quad \langle\omega\rangle = \frac{\int_{-\infty}^{\infty} dt\, f(t)^* i \frac{\partial}{\partial t} f(t)}{\int_{-\infty}^{\infty} dt\, \left|f(t)\right|^2}$$

$$\Delta\omega^2 \equiv \langle[\omega - \langle\omega\rangle]^2\rangle = \frac{\int_{-\infty}^{\infty} dt \left| \left(i\frac{\partial}{\partial t} - \langle\omega\rangle\right) f(t)\right|^2}{\int_{-\infty}^{\infty} dt\, \left|f(t)\right|^2}$$

Take home message:

$$\omega \longleftrightarrow i\frac{\partial}{\partial t}$$

5. We will use a trick which leads to Heisenberg Uncertainty principle in Quantum Mechanics:

Consider a function $\gamma(t)$

$$\begin{aligned}\gamma(\kappa, t) &\equiv \left([t - \langle t\rangle] - i\kappa\left[i\frac{\partial}{\partial t} - \langle\omega\rangle\right]\right) f(t) \\ &= (T - i\kappa\Omega) f(t)\end{aligned}$$

Where κ is a free parameter. Consider:

$$R(\kappa) = \frac{\int_{-\infty}^{\infty} \left|\gamma(\kappa, t)\right|^2 dt}{\int_{-\infty}^{\infty} \left|f(t)\right|^2 dt}$$

Both the numerator and denominator are positive$\longrightarrow R > 0$

$$\begin{aligned}\left|\gamma(\kappa, t)\right|^2 &= (T - i\kappa\Omega) f(t) \cdot (T + i\kappa\Omega^*) f^* \\ &= \left|Tf\right|^2 + \left|\Omega f\right|^2 + i\kappa[Tf\Omega^* f^* - \Omega f T f^*]\end{aligned}$$

Take the last term and simplify:

$$\begin{aligned}& i\kappa\left[Tf\left(-i\frac{\partial}{\partial t} - \cancel{\langle\omega\rangle}\right) f^* - \left(i\frac{\partial}{\partial t} - \cancel{\langle\omega\rangle}\right) f T f^*\right] \\ &= \kappa T\left[f\frac{\partial f^*}{\partial t} + \frac{\partial f}{\partial t} f^*\right] \\ &= \kappa T \frac{\partial}{\partial t}(f f^*)\end{aligned}$$

Now integrate over all time:

$$\begin{aligned}&= \int_{-\infty}^{\infty} dt\, \kappa T \frac{\partial}{\partial t}(f f^*) \\ &= \kappa \underbrace{T f f^*\Big|_{-\infty}^{\infty}}_{=0} - \kappa \int_{-\infty}^{\infty} |f|^2 \underbrace{\frac{\partial T}{\partial t}}_{=1} dt \\ &= -\kappa \int_{-\infty}^{\infty} dt\, |f|^2\end{aligned}$$

Where in the second step we have assumed f is localized such that $f(t=\pm\infty)=0$

(1):

$$\Delta t^2 = \frac{\int_{-\infty}^{\infty} [t-\langle t\rangle]^2 \left|f(t)\right|^2 dt}{\int_{-\infty}^{\infty} \left|f(t)\right|^2 dt}$$

(2):

$$\kappa^2 \Delta\omega^2 = \frac{\kappa^2 \int_{-\infty}^{\infty} \left|\left[i\frac{\partial}{\partial t} - \langle\omega\rangle\right] f(t)\right|^2 dt}{\int_{-\infty}^{\infty} \left|f(t)\right|^2 dt}$$

(3.) using the result from above

$$-\kappa = \frac{\kappa \int_{-\infty}^{\infty} |f(t)|^2 dt}{\int_{-\infty}^{\infty} \left|f(t)\right|^2 dt}$$

$$\Rightarrow \; R = \Delta t^2 + \kappa^2 \Delta\omega^2 - \kappa > 0$$

We know $\dfrac{dR}{d\kappa} = 0$. $R(\kappa)$ minimize at $\kappa = \kappa_{min} = \dfrac{1}{2\Delta\omega^2}$

$$\Rightarrow \; R(\kappa_{min}) = \Delta t^2 - \frac{1}{4\Delta\omega^2} \geq 0$$

Rearranging, we get the Uncertainty principle!!!

$$\Delta t \Delta\omega \geq \frac{1}{2}$$

AM station broadcast: Bandwidth:

$$\Delta\omega = 2\pi\Delta\nu \approx 3 \times 10^4 s^{-1}$$

$\Rightarrow$ Δt is a few $\times 10^{-5}$ seconds. This means we cannot tell signals apart when $\Delta t < O(10^{-5}s)$

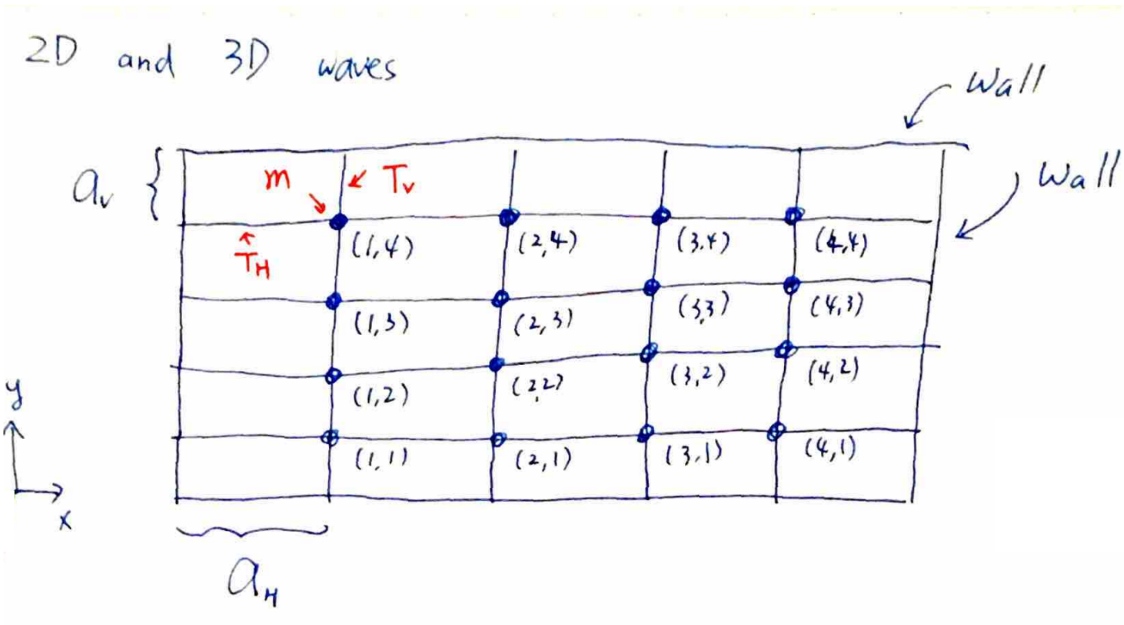
Quantum Mechanics:
We can rewrite Uncertainty as:

$$v\Delta t \cdot \Delta\omega / v \geq 1/2$$
$$\Rightarrow \; \Delta x \cdot \Delta k \geq 1/2$$

In quantum mechanics, $p = \hbar k$ where Planck's Constant $\hbar \equiv h/2\pi$. $\hbar \approx 6.6\times10^{-10}$ eV·s or 1×10^{-34} J·s. We can rewrite our uncertainty principle as:

$$\Rightarrow \ \Delta x \Delta p \geq \frac{\hbar}{2}$$

This is the mathematical statement of the fact that the position and momentum of a particle cannot be specified simultaneously.



Let's move to two and three dimensional waves. In this example, we have a beaded mesh. Index: (j_x, j_y) where each one runs from 1 to 4. First we consider an infinite system by removing the wall and use space translation symmetry:

X direction: Eigenstate of $M^{-1}k$ is $e^{ik_x x}$

Y direction: Eigenstate of $M^{-1}k$ is $e^{ik_y y}$

$$\begin{aligned}\psi(x,y) &= Ae^{ik_x x}e^{ik_y y}\\ &= Ae^{i\vec{k}\cdot\vec{r}}\end{aligned}$$

Where $\vec{k} \equiv k_x\hat{x} + k_y\hat{y}$ and $\vec{r} \equiv x\hat{x} + y\hat{y}$

Time dependent displacement:

$$\psi(x,y,t) = \mathrm{Re}\left[Ae^{i\vec{k}\cdot\vec{r}}e^{-i\omega t}\right]$$

$$\omega^2 = \frac{4T_H}{ma_H}\sin^2\frac{k_x a_H}{2} + \frac{4T_V}{ma_V}\sin^2\frac{k_y a_V}{2}$$

Where k_x an k_y are arbitrary for the moment. In general the two dimensional problem can be infinitely hard!! (But in this special case it is solvable). Before introducing the boundary (infinitely long system)

1. Normal mode in 1D: always two normal modes:

$$e^{\pm ikx}$$

2. But in 2D, in general, a fixed ω gives an infinite number of solutions (if we lower k_x, we can always increase k_y to compensate!)

Adding walls back in:
Boundary conditions give:

$$k_x = \frac{n_x \pi}{L_H} \quad L_H = 5a_H$$
$$k_y = \frac{n_y \pi}{L_V} \quad L_V = 5a_V$$
$$\psi(x,y)_{n_x,n_y} = A_{n_x,n_y} \sin\left(\frac{n_x \pi x}{L_H}\right) sin\left(\frac{n_y \pi y}{L_V}\right)$$

General solution: linear combination of ψ_{n_x,n_y}