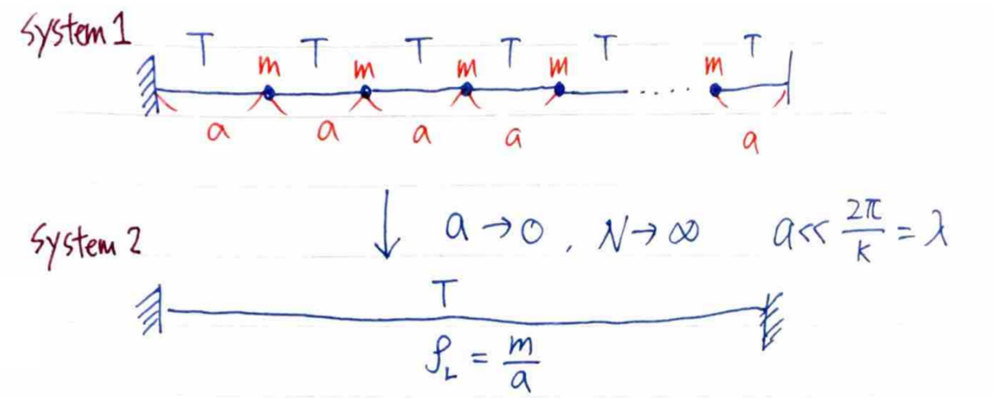
8.03 Lecture 9

Last time:



$$(1): -\ddot{x} = M^{-1}kx \quad M^{-1}kA = \omega^2 A$$

jth term of $M^{-1}kA$: $\omega^2 A_j = \frac{T}{ma}(-A_{j-1} + 2A_j - A_{j+1})$

In the continuum limit: $\omega^2 A(x) = \frac{T}{ma}(-A(x-a) + 2A(x) - A(x-a))$

In the Taylor series:

$$\approx \frac{T}{ma}\left(-\frac{\partial^2 A(x)}{\partial x^2}a^2\right)$$

$$(2): = -\frac{T}{\rho_L}\frac{\partial^2 A(x)}{\partial x^2}$$

$$\Rightarrow \quad M^{-1}k \to -\frac{T}{\rho_L}\frac{\partial^2}{\partial x^2} \text{ and } \psi_j \to \psi(x,t)$$

From (1) and (2):

$$\Rightarrow \frac{\partial^2 \psi(x,t)}{\partial t^2} = \frac{T}{\rho_L}\frac{\partial^2 \psi(x,t)}{\partial x^2}$$

Original dispersion relation:

$$\omega^2 = 4\frac{T}{ma}\sin^2(ka/2)$$

From the fact that $a << 2\pi/k \Rightarrow ka$ is very small.

$$\omega^2 \approx \frac{4T}{ma}\left(\frac{ka}{2}\right)^2 = \frac{T}{\rho_L}k^2$$

$$v_p = \frac{\omega}{k} = \sqrt{\frac{T}{\rho_L}}$$

$$\Rightarrow \frac{\partial^2 \psi(x,t)}{\partial t^2} = v_p^2 \frac{\partial^2 \psi(x,t)}{\partial x^2}$$

The last equation is known as the "wave equation." We get an infinite number of coupled equations of motion. Come back to the original question: What are the normal modes?

$$\psi(x,t) = A(x)B(t)$$

We separate $\psi(x,t)$ into a function that controls the time evolution and a different function that controls the amplitude. Plugging our new ψ into the wave equation:

$$A(x)\frac{\partial^2 B(t)}{\partial t^2} = v_p^2 B(t)\frac{\partial^2 A(x)}{\partial x^2}$$
$$\frac{1}{v_p^2 B(t)}\frac{\partial^2 B(t)}{\partial t^2} = \frac{1}{A(x)}\frac{\partial^2 A(x)}{\partial x^2}$$

This equation must be satisfied for all x and t and so both sides must be equal to a constant. (If this is unfamiliar, think about varying x without varying t; the only way the two sides stay equal is if they are constant.) Now we have:

$$\frac{1}{v_p^2 B(t)}\frac{\partial^2 B(t)}{\partial t^2} = \frac{1}{A(x)}\frac{\partial^2 A(x)}{\partial x^2} = -k_m^2$$

Solving the left hand side first:

$$\frac{1}{v_p^2 B(t)}\frac{\partial^2 B(t)}{\partial t^2} = -k_m^2$$
$$\frac{\partial^2 B(t)}{\partial t^2} = -k_m^2 v_p^2 B(t)$$
$$\Rightarrow \quad B(t) = B_m \sin(\omega_m t + \beta_m)$$

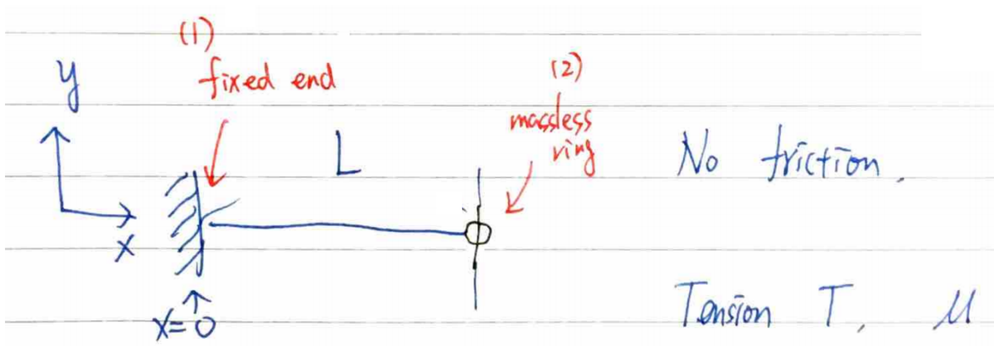
Where $\omega_m \equiv v_p k_m$. Moving to the right hand side:

$$\frac{1}{A(x)}\frac{\partial^2 A(x)}{\partial x^2} = -k_m^2$$
$$\Rightarrow \quad A(t) = C_m \sin(k_m x + \alpha_m)$$

We now have an expression for the mth normal mode:

$$\psi_m(x,t) = A_m \sin(\omega_m t + \beta_m)\sin(k_m x + \alpha_m)$$

$\omega_m = v_p k_m$ is decided by the properties of the string. The two unknowns, α_m and k_m, are decided by the boundary conditions. A_m, β_m are decided by the initial conditions. (Shown later).
*Look at the structure of this normal mode solution. Let's stop and think about what we have learned:
(1) Each point mass on the string is oscillating harmonically (only up and down; not in the horizontal direction!) at the same frequency and phase!
(2) Their relative amplitude: sine function! (The same as the discrete system)
Need to determine the unknown coefficients step by step. Let's take a concrete example: suppose we have a string, one end is fixed and the other end is open.

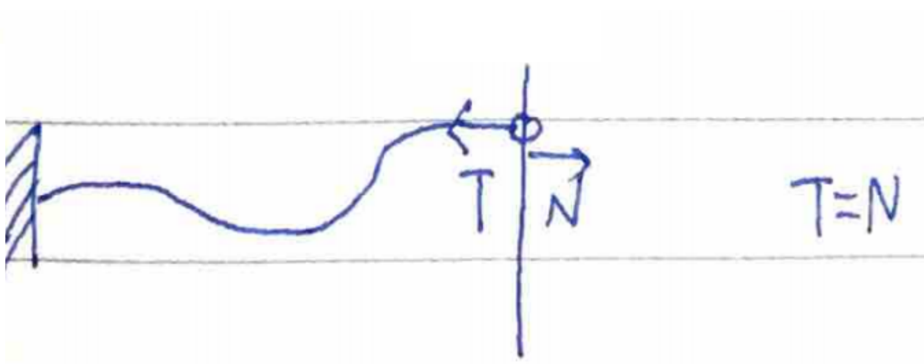


Boundary conditions:

$$(1)\quad x = 0 \;\Rightarrow\; \psi(0,t) = 0$$

$$(2)\quad x = L \;\Rightarrow\; \frac{\partial \psi}{\partial x}(L,t) = 0$$

If $\frac{\partial \psi(L,t)}{\partial x} \neq 0$ then there is a net force (the tension does not cancel with the normal force).



What are the normal modes?

$$(1)\qquad \Rightarrow\; \psi_m(0,t) = A_m \sin(\alpha_m)\sin(\omega_m t + \beta_m) = 0$$
$$\Rightarrow \alpha_m = 0$$

$$(2)\qquad \Rightarrow\; \frac{\partial \psi_m}{\partial x} = A_m k_m \sin(\omega_m t + \beta_m)\cos(k_m x + \alpha_m)$$

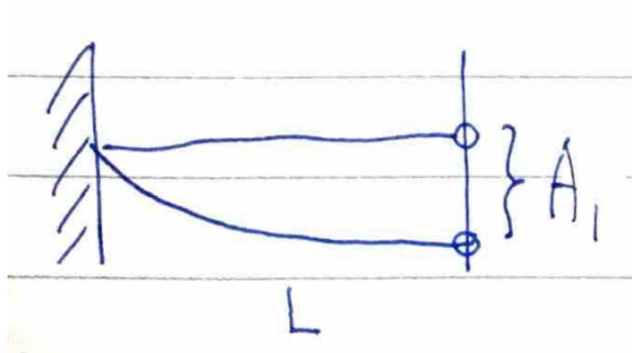
$$\text{At } x = L: \; \frac{\partial \psi_m(L,t)}{\partial x} = 0 = A_m k_m \sin(\omega_m t + \beta_m)\cos(K_m L)$$

$$\Rightarrow k_m L = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$$

$$k_m = \frac{(2m-1)\pi}{2L}$$

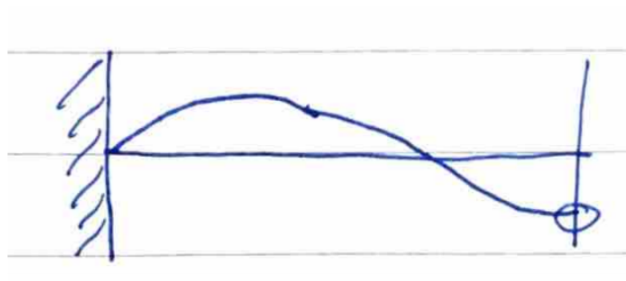
For the first mode, $m = 1$:

$$k_1 = \frac{\pi}{2L} \quad \lambda_1 = \frac{2\pi}{k_1} = 4L \quad \omega_1 = vk_1 = \sqrt{\frac{T}{\mu}}\frac{\pi}{2L}$$



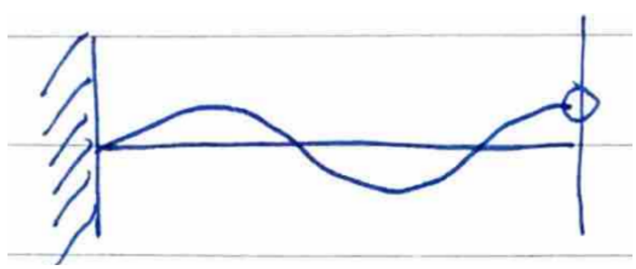
The second mode, $m = 2$:

$$k_2 = \frac{3\pi}{2L} \qquad \lambda_2 = \frac{4}{3}L$$



The third mode, $m = 3$:

$$k_3 = \frac{5\pi}{2L} \qquad \lambda_3 = \frac{4}{5}L$$



The general solution:

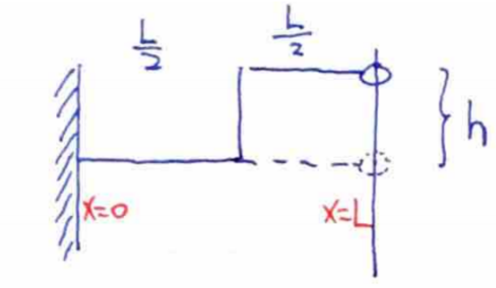
$$\psi(x,t) = \sum_{m=1}^{\infty} A_m \sin(\omega_m t + \beta_m)\sin(k_m x + \alpha_m)$$

From the boundary conditions:

$$\alpha_m = 0 \qquad k_m = \frac{(2m-1)\pi}{2L}$$

$$\psi(x,t) = \sum_{m=1}^{\infty} A_m \sin\left[\frac{(2m-1)v\pi}{2L}t + \beta_m\right] \sin\left[\frac{(2m-1)\pi}{2L}x+\right]$$

How do we extract A_m and β_m?



Suppose at $t = 0$ the string looks like this. Also, the string is at rest.

Initial conditions: (a) $\dot{\psi}(x,0) = 0$ and (b) $\psi(x,0)$ is known.
From (a) we get:

$$\dot{\psi}(x,t) = \sum_{m=1}^{\infty} A_m \omega_m \cos(\omega_m t + \beta_m) \sin(k_m x + \alpha_m)$$

$$\dot{\psi}(x,t) = 0 \Rightarrow \beta_m = \frac{\pi}{2} \Rightarrow \psi(x,0) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{(2m-1)\pi}{2L}x\right)$$

(b) How do I extract A_m from the given $\psi(x,0)$? Use the "orthogonality" of the sine functions:

$$\int_0^L \sin(k_m x)\sin(k_n x)dx = \begin{cases} \frac{L}{2} \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases} \tag{1}$$

We can extract A_m by:

$$A_m = \frac{2}{L}\int_0^L \psi(x,0)\sin(k_m x)dx$$

In this example:

$$\begin{aligned} A_m &= \frac{2}{L}\int_{L/2}^{L} h\sin(k_m x)dx \\ &= \frac{2}{L}\frac{-h}{k_m}\left[\cos(k_m L) - \cos(k_m \frac{L}{2})\right] \end{aligned}$$

Where

$$k_m = \frac{(2m-1)\pi}{2L}$$