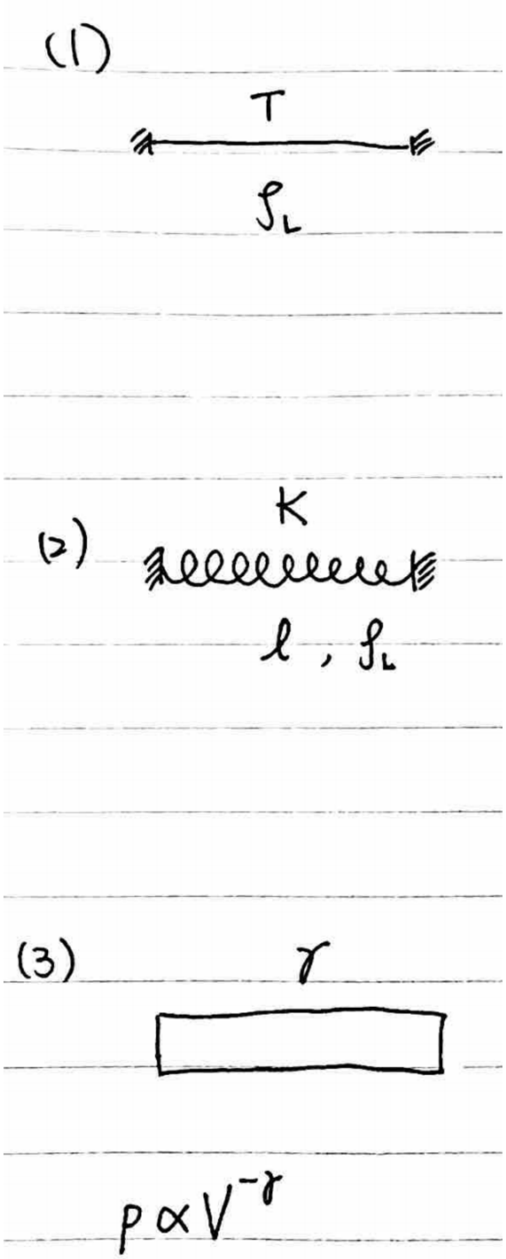
# 8.03 Lecture 12

Systems we have learned:
Wave equation:

$$\frac{\partial^2 \psi}{\partial t^2} = v_p^2 \frac{\partial^2 \psi}{\partial x^2}$$

There are three different kinds of systems discussed in the lecture:




(1) String with constant tension and mass per unit length $\rho_L$

$$v_p = \sqrt{\frac{T}{\rho_L}}$$

(2) Spring with spring constant $k$, length $l$, and mass per unit length $\rho_L$

$$v_p = \sqrt{\frac{kl}{\rho_L}}$$

(3) Organ pipe with room pressure $P_0$ and air density $\rho$

$$v_p = \sqrt{\frac{\gamma P_0}{\rho}}$$

This time, we are doing EM (electromagnetic) waves!

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} && \Rightarrow \text{Gauss' Law} \\
\vec{\nabla} \cdot \vec{B} &= 0 && \Rightarrow \text{Gauss' Law for magnetism} \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} && \Rightarrow \text{Faraday's Law} \\
\vec{\nabla} \times \vec{B} &= \mu_0 \left( \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) && \Rightarrow \text{Ampere's Law}
\end{aligned}
$$

In the vacuum: $\rho = 0$ and $\vec{J} = 0$ and we get:

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= 0 \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \times \vec{B} &= \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}
\end{aligned}
$$

Where in the last two equations we see a changing magnetic field generates an electric field and a changing electric field generates a magnetic field. Can you see the EM wave solution from these equations? Maxwell saw it!
We need to use this identity:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{A}$$

Where $\vec{\nabla} \cdot \vec{\nabla} \equiv \vec{\nabla}^2$ is the Laplacian operator. In the vacuum:

$$\vec{\nabla} \times \cancelto{-\partial \vec{B}/\partial t}{(\vec{\nabla} \times \vec{E})} \quad = \quad \vec{\nabla}\cancelto{0}{(\vec{\nabla} \cdot \vec{E})} - (\vec{\nabla}^2)\vec{E}$$

Where we have made replacements based on the vacuum Maxwell equations above. Let's first examine the left hand side:

$$
\begin{aligned}
\vec{\nabla} \times \left( -\frac{\partial \vec{B}}{\partial t} \right) &= -\frac{\partial}{\partial t} \left( \vec{\nabla} \times \vec{B} \right) \\
&= -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \\
&= -\vec{\nabla}^2 E \\
\Rightarrow \quad \vec{\nabla}^2 E &= \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}
\end{aligned}
$$

Recall

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

And so we have a wave equation!!

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

This equation changed the world! Maxwell is the first one who recognized it because of the term he put in. It was a wave equation with speed equal to the speed of light:

$$v_p = c = \frac{1}{\sqrt{\mu_0\epsilon_0}} \approx 3\cdot 10^8\ m/s$$

What about the $\vec{B}$ field? We can do the same exercise:

$$\vec{\nabla}^2 B = \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

It is very important that the associated magnetic field also satisfies the wave equation. From the Maxwell equation $\vec{E}$ creates $\vec{B}$ and $\vec{B}$ creates $\vec{E}$, therefore they can not exist without each other.

1638 Galileo: speed of light is large

1676 Romer: $2.2\times 10^8\ m/s$

1729 James Bradley: $3.01\times 10^8\ m/s$

This means that in vacuum you can excite EM waves! What is oscillating? The field!
Before we tackle EM waves, let's review divergence and curl briefly.
*Field:
Scalar field: every positing in the space gets a number. Temperature is an example.
Vector field: Instead of a number or scalar, every point gets a vector.

$$\vec{A}(x,y,z) = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

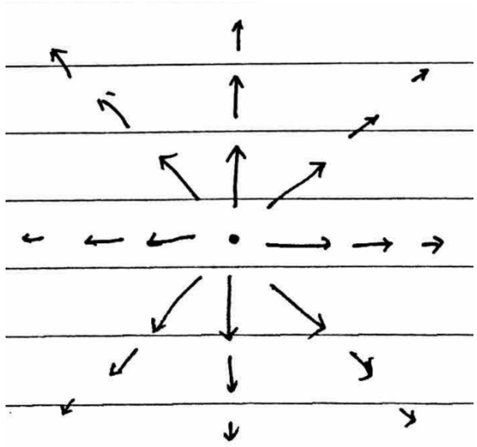
The electric and magnetic fields are vector fields, e.g.:

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

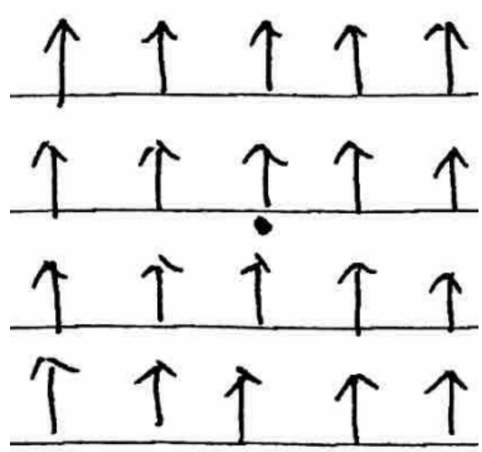
To understand the structure of vector fields:
**Divergence** (using our definition of $\vec{\nabla}$ from above):

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The divergence is a measure of how much the vector $v$ spreads out (*diverges*) from a point:



The divergence of this vector field is positive.

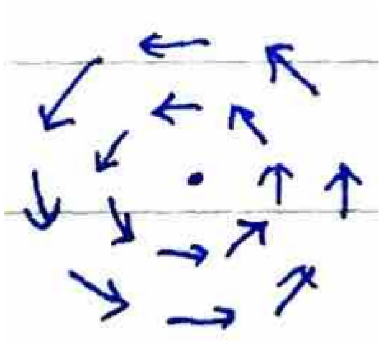


The divergence of this vector field is zero.

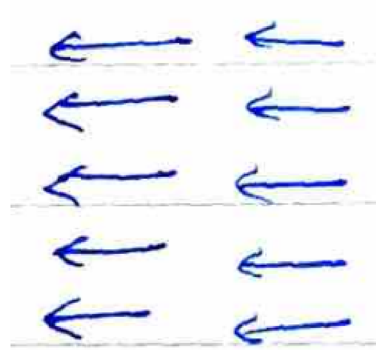
**Curl**:

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z}$$

What exactly does curl mean? It is a measure of how much the vector $\vec{A}$ "curls around" a point.



This vector field has a large curl.



This vector field has no curl.

<u>Gauss' Theorem</u> (or the Divergence Theorem):

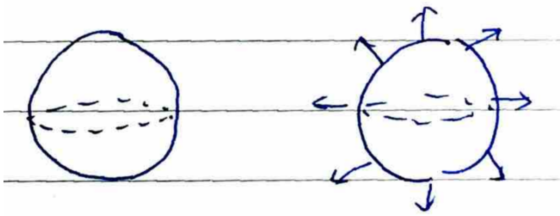
$$\iiint_V \left( \vec{\nabla} \cdot \vec{A} \right) d\tau = \oint_S \vec{A} \cdot \vec{da}$$

Which allows us to relate the integral of the divergence over the whole volume (RHS) to a 2-D surface integral (LHS).

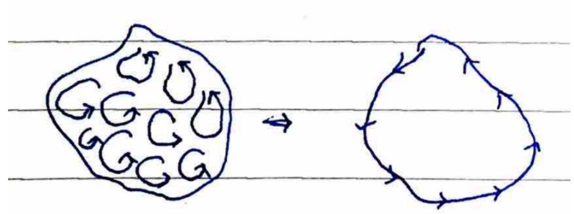
<u>Stokes' Theorem</u>:

$$\iint_S \left( \vec{\nabla} \times \vec{A} \right) \cdot \vec{da} = \oint_P \vec{A} \times \vec{dl}$$

Which allows us to related the surface integral over the curl (LHS) to a line integral integral over a closed path (RHS).



Gauss' Theorem



Stokes' theorem

*Consider a "plane wave" solution:

$$\vec{E} = \text{Re}\left[E_0 e^{i(kz-\omega t)}\hat{x}\right] \quad \text{Only in the } \hat{x} \text{ direction.}$$
$$= \{E_0 \cos(kz-\omega t)\ ,\ 0\ ,\ 0\}$$

Check if it satisfies

$$\vec{\nabla}^2 E = \mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \quad \frac{\partial^2 E_x}{\partial z^2}\hat{x} = \mu_0\epsilon_0 \frac{\partial^2 \vec{E}_x}{\partial t^2}\hat{x}$$

In $\hat{x}$ direction: $-E_0 k^2 \cos(kz-\omega t) = -\mu_0\epsilon_0\omega^2 E_0 \cos(kz-\omega t)$

$$\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c \quad \Rightarrow \quad \text{Condition needed to satisfy the wave equation.}$$

*What about $\vec{B}$?

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \frac{\partial E_x}{\partial z}\hat{y} - \cancelto{0}{\frac{\partial E_x}{\partial y}}\hat{z}$$
$$= -kE_0 \sin(kz-\omega t)\hat{y}$$

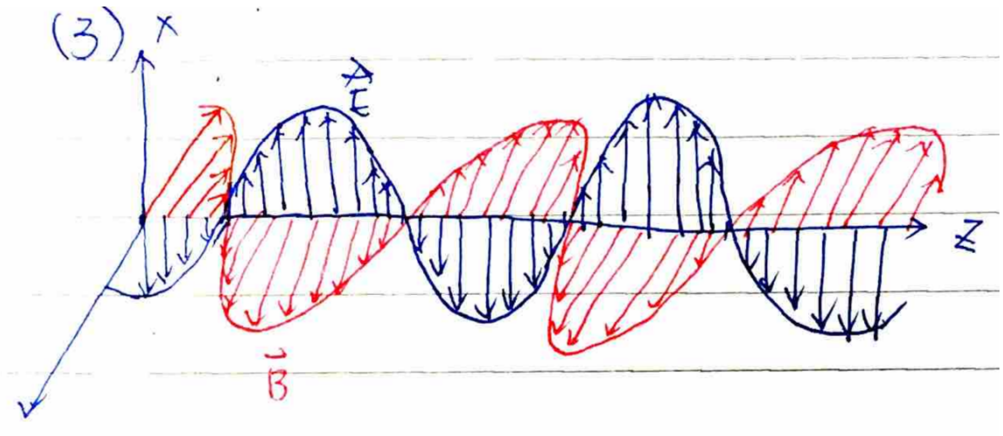
$$\Rightarrow \quad \vec{B} = \frac{k}{\omega}E_0\cos(kz-\omega t)\hat{y} = \frac{E_0}{c}\cos(kz-\omega t)\hat{y}$$

What did we learn from this exercise?

1. $\vec{E}$ must come with $\vec{B}$. In vacuum: the two fields are perpendicular and they are in phase. If $\vec{k}$ is the direction of propagation then $\vec{B} = \frac{1}{c}\hat{k} \times \vec{E}$ The amplitude of the magnetic field is equal to the amplitude of the electric field divided by the speed of light.

2. The EM wave is non-dispersive, meaning that the speed of the wave $c$ is independent of the wave number $k$: $\frac{\omega}{k} = c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$

3. The direction of the propagating EM wave is $\vec{E} \times \vec{B}$




In general a propagating EM wave can be written as:

$$\vec{E}(r,t) = \text{Re}\left[\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t+\phi)}\right]$$

Where $\vec{E}_0 \equiv E_{0_x}\hat{x} + E_{0_y}\hat{y} + E_{0_z}\hat{z}$ , $\vec{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$ and $\omega \equiv ck$
Given this electric field, we can get the magnetic field:

$$\vec{B}(r,t) = \frac{1}{c}\hat{k} \times \vec{E}$$

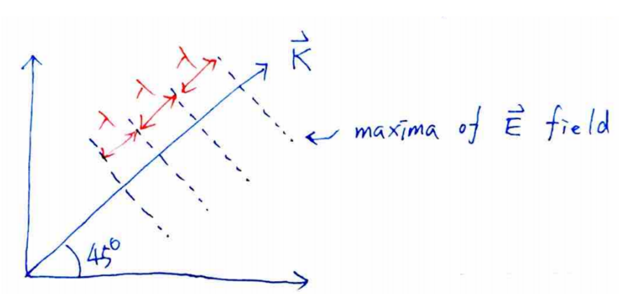
Example:

$$\vec{k} = \frac{2\pi}{\lambda}\left\{\frac{\hat{x}}{\sqrt{2}} + \frac{\hat{y}}{\sqrt{2}}\right\}$$

$$\vec{E}_0 = -\frac{E_0}{\sqrt{2}}\hat{x} + \frac{E_0}{\sqrt{2}}\hat{y}$$

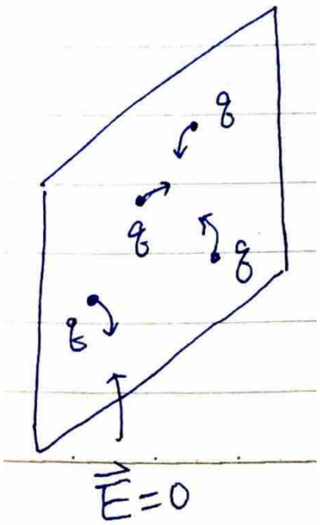
$$\vec{k}\cdot\vec{r} = \frac{2\pi}{\sqrt{2}\lambda}(x+y)$$

$$\Rightarrow \quad \vec{E}(x,y,z) = E_0\left(-\frac{\hat{x}}{\sqrt{2}} + \frac{\hat{y}}{\sqrt{2}}\right)\cos\left(\frac{\sqrt{2}\pi}{\lambda}(x+y) - \omega t\right)$$

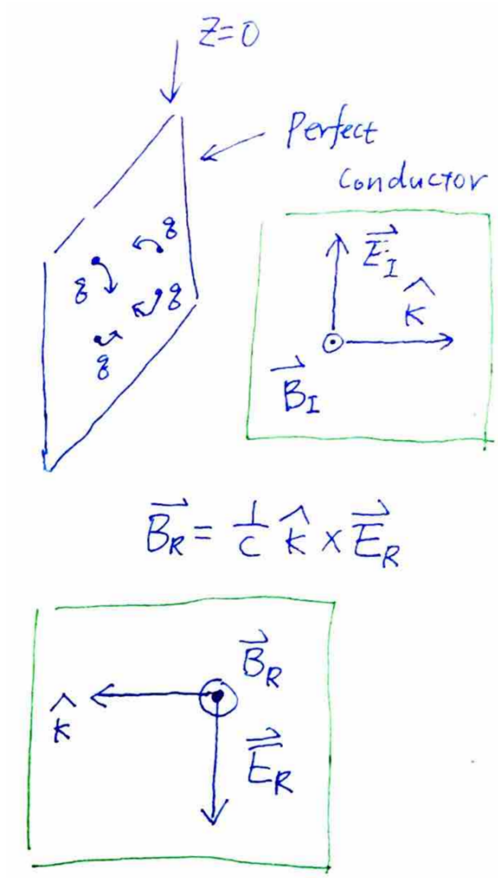


$$\vec{B} = \frac{1}{c}\hat{k} \times \vec{E} \;\Rightarrow\; \vec{B}(x, y, z) = \frac{E_0}{c}\hat{z}\cos\left(\frac{\sqrt{2}\pi}{\lambda}(x + y) - \omega t\right)$$

If there is no other material, this EM wave will travel forever...
Now let's put something into the game: A "perfect conductor"




A busy world inside this system! All the little charges are moving around without cost of energy (there is no dissipation).



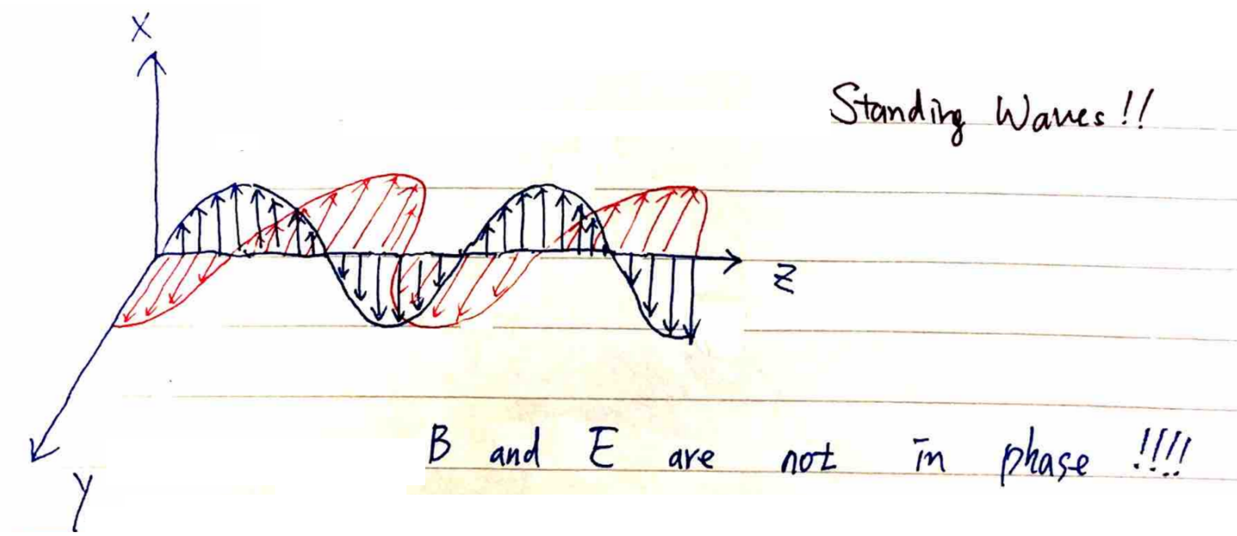

Incident wave:

$$\begin{cases} \vec{E}_I = \frac{E_0}{2}\cos(kz - \omega t)\hat{x} \\ \vec{B}_I = \frac{E_0}{2c}\cos(kz - \omega t)\hat{y} \end{cases}$$

To satisfy the boundary conditions $\vec{E} = 0$ at $z = 0$ we need a reflected wave!

$$\vec{E}_R = -\frac{E_0}{2}\cos(-kz - \omega t)\hat{x}$$
$$\vec{B}_R = \frac{E_0}{2c}\cos(-kz - \omega t)\hat{y}$$

$$\begin{aligned}\vec{E} = \vec{E}_I + \vec{E}_R &= \frac{E_0}{2}\left(\cos(kz-\omega t) - \cos(-kz-\omega t)\right)\hat{x} \\ &= E_0 \sin(\omega t)\sin(kz)\hat{x} \\ \vec{B} = \vec{B}_I + \vec{B}_R &= \frac{E_0}{2c}(\cos(kz-\omega t) + \cos(-kz-\omega t))\hat{y} \\ &= \frac{E_0}{c}\cos(\omega t)\cos(kz)\hat{y}\end{aligned}$$




Energy density?

$$\begin{aligned}U_E &= \frac{1}{2}\epsilon_b E^2 = \frac{\epsilon_0}{2}E_0^2 \sin^2\omega t \sin^2 kz \\ U_B &= \frac{1}{2\mu_0}B^2 = \frac{\epsilon_0}{2}E_0^2 \cos^2\omega t \cos^2 kz\end{aligned}$$

**Poynting vector**: directional energy flux, or the rate of energy transfer per unit area:

$$\begin{aligned}\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} &= \frac{1}{\mu_0}E_x B_y \hat{z} \\ &= \frac{E_0^2}{\mu_0 c}\sin\omega t \cos\omega t \sin kz \cos kz \hat{z} \\ &= \frac{E_0^2}{4\mu_0 c}\sin(2\omega t)\sin(2kz)\hat{z}\end{aligned}$$

This is how a microwave oven works!

*The EM waves are bounced around inside the oven
*EM waves increase the vibration of the molecules in the oven and increase the temperature of the food.