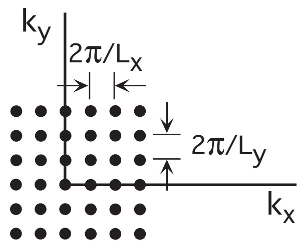
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.044 Statistical Physics I — Spring Term 2013

Solutions, Problem Set #11

Problem 1: Ripplons

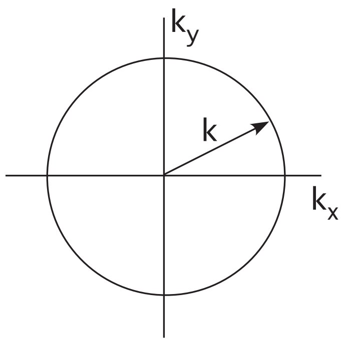
a)



$$\text{k-volume/point} = \frac{(2\pi)^2}{L_x L_y}$$

$$\text{points/k-volume} \equiv D(k) = \frac{L_x L_y}{(2\pi)^2} = \underline{\frac{A}{(2\pi)^2}}$$

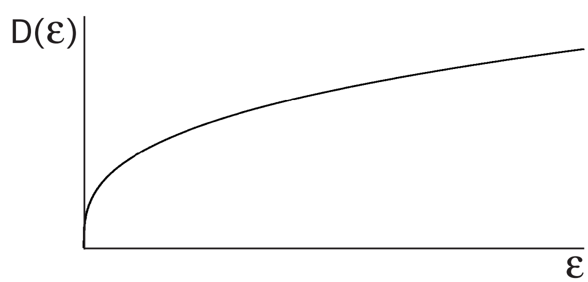
b)



$$\#(\epsilon) = \pi k^2(\epsilon) D(k) \qquad \epsilon = b\, k^{3/2}$$

$$= \pi \left(\frac{\epsilon}{b}\right)^{4/3} D(k)$$

$$D(\epsilon) = \frac{d\#}{d\epsilon} = \frac{4}{3}\frac{A}{(2\pi)^2}\pi\frac{\epsilon^{1/3}}{b^{4/3}}m = \underline{\frac{A}{3\pi b^{4/3}}\,\epsilon^{1/3}}$$



c)

$$U = \int_0^\infty \epsilon\,(< n > + \tfrac{1}{2})D(\epsilon)\,d\epsilon = \int_0^\infty \epsilon\,\left(\frac{1}{e^{\epsilon/k_BT}-1} + \tfrac{1}{2}\right) D(\epsilon)\,d\epsilon$$

$$C_A = \left(\frac{\partial U}{\partial T}\right)_A = \int_0^\infty \epsilon\,\left(\frac{(\epsilon/k_BT^2)e^{\epsilon/k_BT}}{(e^{\epsilon/k_BT}-1)^2} + \tfrac{1}{2}\right)\frac{A}{3\pi b^{4/3}}\,\epsilon^{1/3}\,d\epsilon$$

$$= \underline{\frac{Ak_B}{3\pi b^{4/3}}(k_BT)^{4/3}\int_0^\infty \frac{x^{7/3}e^x}{(e^x-1)^2}\,dx \quad \propto \quad T^{4/3}}$$

One could also proceed directly from U.

$$
\begin{aligned}
U &= \frac{A}{3\pi b^{4/3}} \int_0^\infty \frac{\epsilon^{4/3}}{e^{\epsilon/k_BT} - 1}\, d\epsilon + C \\
&= \frac{A(k_BT)^{7/3}}{3\pi b^{4/3}} \int_0^\infty \frac{x^{4/3}}{e^x - 1}\, dx + C \\
C_A &= \left(\frac{\partial U}{\partial T}\right)_A = \underline{\frac{7Ak_B}{9\pi b^{4/3}}\, (k_BT)^{4/3} \int_0^\infty \frac{x^{4/3}}{e^x - 1}\, dx \quad \propto \quad T^{4/3}}
\end{aligned}
$$

d) There is no energy gap behavior because there is no energy gap. For any k_BT there are always oscillators with $\hbar\omega < k_BT$.

Problem 2: Two-Dimensional Metal

a)

$$e^{ik_x(x+L)} = e^{ik_xx}\, e^{ik_xL} = e^{ik_xx} \Rightarrow k_x = n_x \frac{2\pi}{L} \qquad n_x = \pm 1, \pm 2, \cdots$$

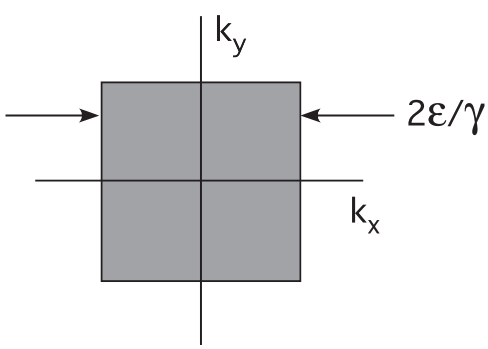
The same holds of k_y.

$$\underline{\vec{k} = \frac{2\pi}{L}(n_x\hat{x} + n_y\hat{y}) \qquad n_i = \pm 1, \pm 2, \cdots}$$

b)

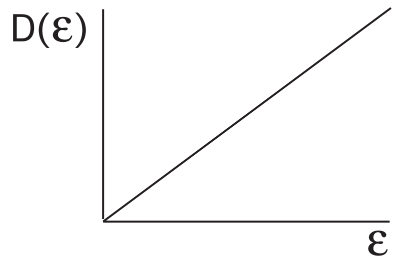
$$\underline{D(\vec{k}) = \left(\frac{L}{2\pi}\right)^2 \quad \text{for all } \vec{k}}$$

c)



Taking into account the spin degeneracy

$$
\begin{aligned}
\#(\epsilon) &= 2D(\vec{k}) \left(\frac{2\epsilon}{\gamma}\right)^2 \\
D(\epsilon) &= 2\left(\frac{L}{2\pi}\right)^2 \frac{8}{\gamma^2}\, \epsilon \\
&= \underline{4\left(\frac{L}{\pi\gamma}\right)^2 \epsilon}
\end{aligned}
$$



d)

$$N = \#(\epsilon_F) = \frac{8L^2}{(2\pi)^2}\left(\frac{\epsilon}{\gamma}\right)$$
$$\epsilon_F = \underline{\left(\tfrac{1}{2}\,\pi^2\gamma^2(N/A)\right)^{1/2}}$$

e)

$$D(\epsilon) = a\,\epsilon \quad N = \int_0^{\epsilon_F} a\,\epsilon\,d\epsilon = \frac{a}{2}\epsilon_F^2$$
$$U = \int_0^{\epsilon_F} a\,\epsilon^2\,d\epsilon = \frac{a}{3}\epsilon_F^3 = \underline{\tfrac{2}{3}N\epsilon_F}$$

f) Thermal agitation only disturbs a fraction k_BT/ϵ_F of the total number of electrons, and imparts to them an energy of the order of k_BT. Thus the total increase in energy from the $T=0$ value is proportional to T^2 and the heat capacity will be linearly proportional to T.

g)

$$dU = \underbrace{TdS}_{0 \text{ at } T=0} + \mathcal{S}dA$$

$$\mathcal{S} = \frac{dU}{dA} = \frac{2}{3}N\frac{d\epsilon_F}{dA} = \frac{2}{3}N(-\frac{1}{2})\frac{\epsilon_F}{A} = \underline{-\frac{1}{3}\left(\frac{N}{A}\right)\epsilon_F}$$

Problem 3: Donor Impurity States in a Semiconductor

a)

$$< n > \;\to\; e^{-(\epsilon-\mu)/k_B T} = e^{-(\Delta-\mu)/k_B T}\, e^{-(\epsilon-\Delta)/k_B T}$$

$$\begin{aligned}
D_{\text{states}}(\epsilon) &= \frac{V}{2\pi^2}\left(\frac{2m_e}{\hbar^2}\right)^{3/2}(\epsilon-\Delta)^{1/2} \\
N_C &= \int_\Delta^\infty < n_e > D(\epsilon)\, d\epsilon \\
&= \frac{V}{2\pi^2}\left(\frac{2m_e}{\hbar^2}\right)^{3/2} e^{-(\Delta-\mu)/k_B T}\int_0^\infty \sqrt{\delta}\, e^{-\delta/k_B T}\, d\delta \\
&= \frac{V}{2\pi^2}\left(\frac{2m_e k_B T}{\hbar^2}\right)^{3/2} e^{-(\Delta-\mu)/k_B T}\underbrace{\int_0^\infty \sqrt{x}\, e^{-x}\, dx}_{\sqrt{\pi}/2} \\
&= \frac{V}{4}\left(\frac{2m_e k_B T}{\pi\hbar^2}\right)^{3/2} e^{-(\Delta-\mu)/k_B T} \\
&= \alpha(T) e^{-\Delta/k_B T} e^{\mu/k_B T} \quad \text{where} \quad \underline{\alpha(T) \equiv \frac{V}{4}\left(\frac{2m_e k_B T}{\pi\hbar^2}\right)^{3/2}}
\end{aligned}$$

b) $< n >$ is equal to $1/2$ where $\epsilon = \mu$, so if the donors are only half occupied, and they are located at $\epsilon = 0$, then $\mu = 0$. Since half of the donor electrons are now in the conduction band, the last equation above becomes

$$N_D/2 = \alpha(T) e^{-\Delta/k_B T}$$

This could be solved numerically for T.

c) When the energy is many times k_BT below the chemical potential μ,

$$< n > \to 1 - e^{(\epsilon-\mu)/k_BT}$$

Then remembering that $\epsilon = 0$ at the position of the donors

$$\begin{aligned}
N_D &= N_D(1 - e^{(-\mu)/k_BT}) + \alpha(T)e^{-\Delta/k_BT}e^{\mu/k_BT} \\
N_D e^{(-\mu)/k_BT} &= \alpha(T)e^{-\Delta/k_BT}e^{\mu/k_BT} \\
N_D/\alpha(T) &= e^{-\Delta/k_BT}e^{2\mu/k_BT} \\
k_BT \ln[N_D/\alpha(T)] &= -\Delta + 2\mu \\
\mu &= \underline{\Delta/2 + (k_BT/2)\ln[N_D/\alpha(T)]}
\end{aligned}$$

Since $\alpha(T) \propto T^{3/2}$ the temperature dependent term above is proportional to

$$T\,[\ln(T^{-3/2}) + \ln(\text{constant})] = -(3/2)T\ln T + T\ln(\text{constant})$$

which goes to zero at $T = 0$. Thus $\epsilon_F = \Delta/2$

d)

$$
\begin{aligned}
< n > &= \frac{1}{\exp[(\epsilon-\mu)/k_BT]+1} \\
N(e \text{ on donor sites}) &= \frac{N_D}{\exp[-\mu/k_BT]+1} \\
N_C &= \alpha(T)\exp[-\Delta/k_BT]\ \exp[\mu/k_BT] \\
\exp[-\mu/k_BT] &= \frac{\alpha(T)}{N_C}\exp[-\Delta/k_BT] \\
N(e \text{ on donor sites}) &= \frac{N_D}{(\alpha/N_C)\exp[-\Delta/k_BT]+1} \\
N(e \text{ on donor sites}) + N_C &= \frac{N_D}{(\alpha/N_C)\exp[-\Delta/k_BT]+1} + N_C = N_D \\
N_D + N_C(\alpha/N_C)\exp[-\Delta/k_BT] + N_C &= N_D(\alpha/N_C)\exp[-\Delta/k_BT] + N_D \\
(\alpha/N_D)\exp[-\Delta/k_BT] + (N_C/N_D) &= (N_D/N_C)(\alpha/N_D)\exp[-\Delta/k_BT] \\
(N_C/N_D)(\alpha/N_D)\exp[-\Delta/k_BT] + (N_C/N_D)^2 &= (\alpha/N_D)\exp[-\Delta/k_BT]
\end{aligned}
$$

$$
\boxed{\left(\frac{N_C}{N_D}\right)^2 + \left(\frac{N_C}{N_D}\right)\frac{\alpha}{N_D}e^{-\Delta/k_BT} - \frac{\alpha}{N_D}e^{-\Delta/k_BT} = 0}
$$

Set $N_C/N_D = 1/2$.

$$
\frac{1}{4} + \frac{1}{2}\frac{\alpha}{N_D}e^{-\Delta/k_BT} - \frac{\alpha}{N_D}e^{-\Delta/k_BT} = 0
$$

$$
\underline{N_D = 2\alpha(T)e^{-\Delta/k_BT}}
$$

When $T \to 0$, $\exp[-\Delta/k_BT] \to 0$ leaving $(N_C/N_D)^2 = 0$, so in this limit $\underline{N_C = 0}$

When $T \to \infty$ $\alpha(T)$ grows without bound and the first term in the equation can be neglected leaving

$$
\left(\frac{N_C}{N_D}\right)\frac{\alpha}{N_D}e^{-\Delta/k_BT} - \frac{\alpha}{N_D}e^{-\Delta/k_BT} = 0
$$

with the solution $\underline{N_C = N_D}$.

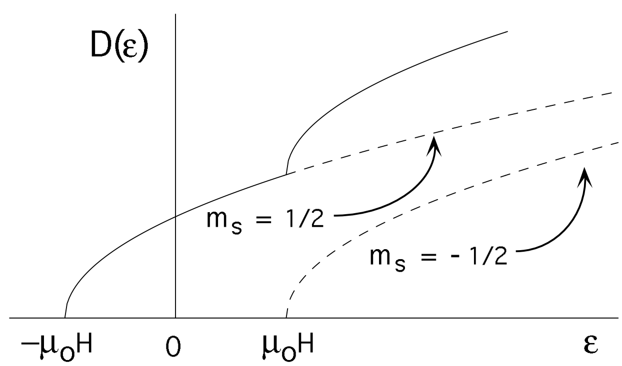
Problem 4: Spin Polarization

a) Recall that in zero field the density of states for spin-$\frac{1}{2}$ Fermions is

$$\begin{aligned} D_0(\epsilon) &= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2} \qquad\qquad \epsilon > 0 \\ &= 0 \qquad\qquad\qquad\qquad\qquad \epsilon < 0 \end{aligned}$$

These are equally divided between spin up and spin down. The application of a field shifts all $m_s = \frac{1}{2}$ states down in energy by $\mu_0 H$ and all $m_s = -\frac{1}{2}$ states up by $\mu_0 H$ (assuming a positive μ_0, that is, a magnetic moment parallel rather than anti-parallel to $\vec{S}$). Thus

$$\begin{aligned} D_{\frac{1}{2}}(\epsilon) &= \frac{1}{2}D_0(\epsilon + \mu_0 H) \\ D_{-\frac{1}{2}}(\epsilon) &= \frac{1}{2}D_0(\epsilon - \mu_0 H) \\ D(\epsilon) &= \frac{1}{2}[D_0(\epsilon + \mu_0 H) + D_0(\epsilon - \mu_0 H)] \end{aligned}$$



b) The filling of $D(\epsilon)$ at $T = 0$ must stop just short of $\epsilon = \mu_0 H$ where the $m_s = -\frac{1}{2}$ states would begin to fill.

$$\begin{aligned} N &= \int_{-\mu_0 H}^{\mu_0 H} D(\epsilon)\, d\epsilon = \int_0^{2\mu_0 H} \tfrac{1}{2}\underbrace{D_0(\epsilon)}_{a\epsilon^{1/2}}\, d\epsilon \\ &= \frac{a}{2}\frac{2}{3}(2\mu_0 H)^{3/2} \\ &= \frac{1}{3}\left[\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\right](2\mu_0 H)^{3/2} \\ 6\pi^2 N/V &= \left(\frac{4m\mu_0 H}{\hbar^2}\right)^{3/2} \end{aligned}$$

$$\begin{aligned}(6\pi^2 N/V)^{2/3} &= \frac{4m\mu_0}{\hbar^2} H \\ H_0 &= \underline{\frac{1}{\mu_0} \frac{\hbar^2 (6\pi^2 N/V)^{2/3}}{4m}}\end{aligned}$$

c) Using cgs units

$$H_0 = 4.22 \times 10^{-54} \frac{1}{\mu_0 M} (N/V)^{2/3}$$

Using $M = 9.11 \times 10^{-28}$g, $\mu_0 = -9.27 \times 10^{-21}$ergs-gauss^{-1}, and $n = 8.45 \times 10^{22}$cm^{-3} gives

$$H_0 = 9.6 \times 10^8 \text{gauss} = \underline{9.6 \times 10^4 \text{Tesla}}.$$

The negative μ_0 means that the electron spins are polarized anti-parallel to the direction of $\vec{H}$.

d) For ^{3}He, $M = 5.01 \times 10^{-24}$g, $\mu_0 = 1.075 \times 10^{-23}$ergs-gauss^{-1}, and $n = 1.64 \times 10^{22}$cm^{-3}. So in this system

$$H_0 = 5.1 \times 10^7 \text{gauss} = \underline{5.1 \times 10^3 \text{Tesla}}.$$

Problem 5: $T = 0$ solubility of ^{3}He in ^{4}He

a) Recall that for non-interacting spin 1/2 Fermions in 3 dimensions with an effective mass m^*

$$\begin{aligned}k_F &= \left(3\pi^2 (N/V)\right)^{1/3} \\ \epsilon_F &= \frac{\hbar^2}{2m^*} \left(\frac{3\pi^2 N}{V}\right)^{2/3}\end{aligned}$$

When the Fermi energy of ^{3}He in ^{4}He equals the potential difference W, any additional ^{3}He atoms will begin to fill their own phase where their kinetic energy can begin at zero. Thus

$$\begin{aligned}W &= \frac{\hbar^2}{2m^*} \left(3\pi^2 n_3^{max}\right)^{2/3} \\ \frac{2m^* W}{\hbar^2} &= \left(3\pi^2 n_3^{max}\right)^{2/3} \\ n_3^{max} &= \underline{\frac{1}{3\pi^2} \left(\frac{2m^* W}{\hbar^2}\right)^{3/2}}\end{aligned}$$

b)

$$
\begin{aligned}
n_3^{max} &= \frac{1}{3\pi^2}\left(\frac{2m^*W}{\hbar^2}\right)^{3/2} \\
&= \frac{1}{3\pi^2}\left(\frac{2\times 10^{-23}\times 0.5\times 10^{-16}}{10^{-54}}\right)^{3/2} \\
&= \frac{1}{3\pi^2}\left(\frac{10^{-39}}{10^{-54}}\right)^{3/2} = \frac{1}{3\pi^2}\left(10\times 10^{14}\right)^{3/2} \\
&= \frac{10\sqrt{10}}{3\pi^2}\times 10^{21} \approx 10^{21}
\end{aligned}
$$

Then

$$\frac{n_3^{max}}{n_3+n_4} \approx \frac{10^{21}}{2\times 10^{22}} = 5\%$$

This is close to the value indicated in the figure.

Problem 6: Melting Curve of ^{3}He

a) The entropy of a Fermi gas at $T = 0$, the model for the liquid ^{3}He, is zero.

b) The entropy of N distinguishable, stationary spin 1/2 particles in the absence of an applied magnetic field is $Nk_B \ln 2$

c) At $T = 0$ in the presence of a large "effective magnetic field" each of the spins in the solid will be oriented with its moment anti-parallel to the effective field. Thus the entropy of the solid goes to $Nk_B \ln 1 =$ zero.

d) In region II the slope of the melting curve is $-k_B \ln 2/v_0$. Yes, it is negative.

e) In region I the slope of the melting curve goes to zero.

f) When heat is added to the system, the entropy must go up. The only way for that to happen is for some of the liquid to turn into solid, that is, to freeze. In this region on the melting curve, when one heats the system it solidifies and when one cools it, it melts.