MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I — Spring Term 2013

Exam #2

Problem 1 (30 points) Entropy of a Surface Film

The surface tension γ and heat capacity at constant area C_A for a water surface of area A covered by a thin film containing N organic molecules are given by

$$\gamma(T, A) = \gamma_0 - \frac{NkT}{A - bN} + \frac{aN^2}{A^2}$$

$$C_A(T) = Nk_B + Nk_B\left(\frac{T}{T_0}\right)^2$$

where a, b, γ_0 and T_0 are constants and k_B is Boltzmann's constant. [Note: Previously we used $\mathcal{S}$ to represent the surface tension but I have changed its symbol here to avoid any confusion with the symbol for the entropy, S.] Expressions for differential work are given on the last page.

Find an expression for the entropy $S(T, A)$ of the surface film, up to an additive constant.

Problem 2 (40 points) Crystal Field Splitting

A solid contains N similar impurity atoms fixed at random lattice sites throughout the crystal. (Because they can not move, they are not considered to be 'identical' particles.) The impurity atoms have an angular momentum $L = 1$. In free space the three levels $m = 1, 0$, and -1 are degenerate. In the crystal the levels are split by the electric fields of the neighboring atoms. The $m = 0$ state has energy 0 and the $m = 1$ and $m = -1$ states have energy Δ.

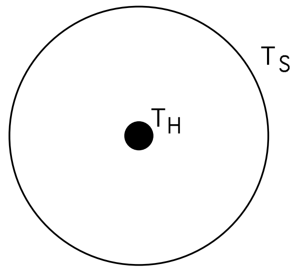
a) (10) Find the contribution of the impurities to the internal energy $U(T, N)$ of the crystal.

b) (10) Without doing any calculations give the impurity contribution to the entropy at $T = 0$ and in the limit where $k_B T \gg \Delta$.

c) (10) Find an expression for the impurity contribution to the heat capacity $C_V(T, N)$. Make a sketch of your result indicating clearly the behavior at high and low temperatures.

d) (10) The crystal field splitting increases when the crystal is compressed. For modest compressions this can be expressed as $d\Delta/dV = -\gamma(\Delta/V)$ where γ is a constant. Using this information, find the spreading pressure $P(T, N)$ exerted on the host crystal lattice by the presence of the impurities. [In checking your answer it may be helpful to note that the units of pressure are energy per unit volume.]

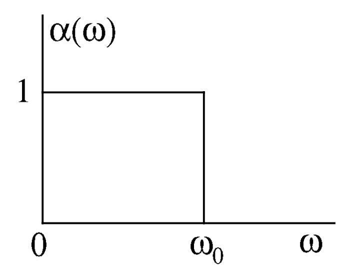
Problem 3 (30 points) Heating a Shell



Somewhere in the vacuum of free space a spherical shell of radius R and absorptivity α is heated from within by a small centered spherical blackbody of radius $r << R$ which is held at a high temperature T_H. The shell loses energy by radiating outward into the vacuum.

a) (10) If $\alpha = 1$ independent of frequency, find the steady state temperature of the shell T_S in terms of T_H, R and r.

Now imagine that the shell absorbs all incident power below ω_0 and is completely transparent above this frequency. Thus α has the frequency dependence shown at the right. $\hbar\omega_0 << kT_H$ so that for the frequencies that get absorbed the universal blackbody energy density expression $u(\omega, T)$ can be approximated as follows.



$$u(\omega, T) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{\exp(\hbar\omega/kT) - 1} \quad \rightarrow \quad \frac{kT\omega^2}{\pi^2 c^3}$$

b) (10) Find the total power absorbed by the shell in terms of k, c, T_H, ω_0, and the appropriate radii.

c) (10) Find the steady state temperature of the shell T_S in terms of T_H, R and r assuming that $\hbar\omega_0 << kT_S$.

Work in simple systems

Hydrostatic system	$-PdV$
Surface film	$\gamma\, dA$
Linear system	$\mathcal{F} dL$
Dielectric material	$\mathcal{E} d\mathcal{P}$
Magnetic material	HdM

Thermodynamic Potentials when work done on the system is $dW = Xdx$

Energy	E	$dE = TdS + Xdx$
Helmholtz free energy	$F = E - TS$	$dF = -SdT + Xdx$
Gibbs free energy	$G = E - TS - Xx$	$dG = -SdT - xdX$
Enthalpy	$H = E - Xx$	$dH = TdS - xdX$

Statistical Mechanics of a Quantum Harmonic Oscillator

$$\epsilon(n) = (n + \tfrac{1}{2})\hbar\omega \qquad n = 0, 1, 2, \ldots$$

$$p(n) = e^{-(n+\frac{1}{2})\hbar\omega/kT}/Z(T)$$

$$Z(T) = e^{-\frac{1}{2}\hbar\omega/kT}(1 - e^{-\hbar\omega/kT})^{-1}$$

$$< \epsilon(n) >= \tfrac{1}{2}\hbar\omega + \hbar\omega(e^{\hbar\omega/kT} - 1)^{-1}$$

Radiation laws

Kirchoff's law: $e(\omega, T)/\alpha(\omega, T) = \frac{1}{4} c\, u(\omega, T)$ for all materials where $e(\omega, T)$ is the emissive power per unit area and per unit frequency interval, $\alpha(\omega, T)$ the absorptivity of the material, and $u(\omega, T)$ is the universal blackbody energy density function.

Stefan-Boltzmann law: $e(T) = \sigma T^4$ for a blackbody where $e(T)$ is the emissive power per unit area integrated over all frequencies. ($\sigma = 56.9 \times 10^{-9}$ watt-m^{-2}K^{-4})

Integrals

$$\int e^{ax}\, dx = \frac{e^{ax}}{a}$$

$$\int x\, e^{ax}\, dx = \frac{e^{ax}}{a^2}(ax - 1)$$

$$\int x^2\, e^{ax}\, dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2)$$

$$\int \frac{dx}{1 + e^x} = \ln\left[\frac{e^x}{1 + e^x}\right]$$

Definite Integrals

For integer n and m

$$\int_0^\infty x^n e^{-x}\, dx = n!$$

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\, dx = \sqrt{\pi}$$

$$(2\pi\sigma^2)^{-1/2} \int_{-\infty}^\infty x^{2n} e^{-x^2/2\sigma^2}\, dx = 1 \cdot 3 \cdot 5 \cdots (2n - 1)\, \sigma^{2n}$$

$$\int_0^\infty x\, e^{-x^2}\, dx = \frac{1}{2}$$

$$\int_0^1 x^m (1 - x)^n\, dx = \frac{n!m!}{(m + n + 1)!}$$

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