

Quantum Physics III (8.06) Spring 2005

Assignment 5

Tuesday March 1, 2005

Due Tuesday March 8, 2005

Readings

The reading assignment for this problem set and part of the next one is:

- Griffiths all of Chapter 6.
- Cohen-Tannoudji Chapter XI including Complements A-D.

Problem Set 5

1. The Fractional Quantum Hall Effect (5 points)

We will not be able to study the fractional quantum hall effect in 8.06. However, I want to give you at least a some sense of it, at a qualitative level. At the very least, I want to convey how the discovery of this effect was even more of a surprise than the discovery of the integer quantum hall effect. This problem is a bona fide part of this problem set, and “counts” in that sense. However, the material alluded to in this problem will not appear on the midterm or final.

The fractional quantum hall effect was discovered in 1982 by Tsui, Störmer and Gossard. They studied a very clean sample of the same sort in which the integer quantum hall effect had been discovered two years earlier. At a low temperature, and in a high enough magnetic field, they discovered a plateau with

$$\frac{1}{R_{\text{hall}}} = \frac{1}{3} \frac{e^2}{h} \quad \text{and} \quad R_{\text{long}} = 0 \quad (1)$$

occurring over a range of filling fractions centered at

$$\frac{nhc}{eB} = \frac{1}{3}.$$

If we attempt to describe this with 8.06 physics, we would say we have a one-third filled Landau level, and as such would have very many degenerate ground states to choose between, corresponding to the choice of which third of the states that make up the lowest Landau level we fill, and which two thirds we leave

empty. Depending on what choice among the many possibilities we made, we could find many different values of R_{hall} and would generically find $R_{\text{long}} \neq 0$. This cannot describe the experimental result.

The correct description of the fractional quantum hall effect relies crucially on the Coulomb interaction between the electrons. The state described by the “Laughlin wave function”, named after Robert Laughlin who proposed it as a way to understand the experimental results, cannot be described by first solving for single electron wave functions, and then filling some of these while leaving others empty. In other words, it is not a direct product of single electron wave functions. It is an intrinsically many-electron wave function, requiring methods beyond those we shall learn in 8.06 to study. However, we can understand at a qualitative level why its discovery required a very clean sample: in order for the fractional quantum hall effect to be observed, the effects of Coulomb repulsion between the electrons must dominate over the interaction between electrons and impurities. In the time since Tsui et al’s discovery, as ever cleaner samples have been studied, plateaus have been seen at more and more fractional hall conductivities, for example at $1/3$, $2/5$, $3/7$, $4/9$, $5/11$, $6/13$, \dots and $2/3$, $3/5$, $4/7$, $5/9$, $6/11$, $7/13$, \dots and many more. It turns out that all these states are described by nondegenerate ground state wave functions, very different from the massively degenerate states we would construct à la 8.06, starting with single electron states, ignoring Coulomb interactions, and simply filling a fraction of a Landau level.

The purpose of this problem is for you to work out one very surprising consequence of the existence of the “ $1/3$ -plateau”. Take the experimental facts (1) at $nhc/eB = 1/3$ as given. Imagine inserting an infinitesimally thin solenoid *within* this sample, perpendicular to the plane of the sample. (The flux tube “pokes through” the sample. This is a thought experiment.) Slowly increase the flux through this solenoid from zero to Φ_0 , the quantum of flux. Now, consider a circular path, within the sample, encircling the flux tube at a great distance. Use the experimentally measured hall conductance to determine how much charge crosses this circular path as the flux in the solenoid increases from 0 to Φ_0 . Call your answer “ Q ”. Determine the magnitude of Q but do not worry about its sign.

You’ve now completed the problem I’ve posed for you, but lets take a minute to understand the significance of the answer. From our study of the Aharonov-Bohm effect, we can conclude that the Hamiltonian of the system is the same at the beginning and the end of the shenanigans you have performed above. Flux 0 and flux Φ_0 in your flux tube give the same Hamiltonian for the electrons in the sample. However, because your flux tube penetrates through the sample, during the time the flux was increasing it can have had a nontrivial effect (beyond just the Aharonov-Bohm effect) on those electrons whose wave functions are nonzero

at the location of the solenoid, and which thus feel the magnetic field in the solenoid. What this means is that although the Hamiltonian at the end of your shenanigans is the same as at the beginning, the state of the system may now be in an excited state, with a different energy. Assuming that the system started in its ground state, it could now be in an excited state. In fact, this is what has happened. By increasing the flux in the solenoid as you did, you have created an excitation of the system, with some finite but nonzero energy, localized in the vicinity of your flux tube. The calculation you did above determines the charge Q of this excitation. This argument, due originally to Laughlin, shows that (regardless of what the detailed description of its ground state wave function turns out to be) the observed $1/3$ quantum hall plateau must be described by a Hamiltonian which includes localized excitations whose charge is Q . This should convince you that its ground state wave function is quite nontrivial! In addition to being nondegenerate, the ground state is also “incompressible”. If you think carefully about the thought experiment you analyzed, you should get a sense of the meaning of this term in this context.

2. A Delta-Function Bump in the Center of an Infinite Square Well (20 points)

Parts of this problem are similar to Griffiths’ problems 6.1 and 6.4.

Suppose that H^0 describes a particle of mass m in an infinite potential well with $V(x) = 0$ for $-a/2 < x < a/2$ and $V = \infty$ elsewhere. Now add a delta-function bump in the center of the infinite square well:

$$H' = \alpha\delta(x)$$

where α is a constant.

- (a) Write down the eigenfunctions $\psi_n^0(x)$ of the unperturbed Hamiltonian H_0 . Make sure that they are properly normalized.
[Note: the notation here is that the ground state of H^0 is the “ $n = 1$ ” state. That is, the unperturbed energies are $E_n^0 = n^2\pi^2\hbar^2/(2ma^2)$ for $n = 1, 2, 3, \dots$]
- (b) Find the first-order correction to the allowed energies. Explain why the energies are not perturbed for even n .
- (c) Write down the expression for the change in the wave function of the ground state due to H_1 to first order in α . Evaluate the integrals that occur, but do not attempt to do the sum.
- (d) Write an expression for the probability to find the particle at $x = 0$ in the ground state, $|\psi_1(0)|^2$, that is accurate to first order in α . Evaluate your result as a function of the parameter $ma\alpha$. Plot $|\psi_1(0)|^2/|\psi_1^0(0)|^2$

versus $m\alpha$ and give a physical reason for the shape of the curve. Give an estimate for the size of α where you would no longer trust first order perturbation theory even qualitatively.

[In this part, you will need to perform an infinite sum. You can compute it numerically, analytically, use Mathematica, or just estimate it.]

- (e) Find the second-order corrections to the energies (ie find E_n^2). For what value of $m\alpha$ is the second order correction to the ground state energy equal to the first order correction? [This is another way of estimating at what point perturbation theory is no longer a reliable guide.]
- (f) This problem can be solved exactly. Define $k_n = \sqrt{2mE_n}/\hbar$ and use 8.04 methods to show that the exact energy eigenvalues E_n must satisfy either $\sin(k_n a/2) = 0$ (for even n) or $\tan(k_n a/2) = -\hbar^2 k_n / (m\alpha)$ (for odd n). Discuss these exact results in the limit of small α , and show that the first order order approximation to the energies that you obtained in part (b) is in fact the first order approximation to the exact energies.

[Optional: show that the second order approximation you obtained in part (e) is the second order approximation to the exact energies.]

[Even more optional: in parts (c) and (d), you found $\psi_1(x)$ to order α . Now, find the exact ground state wave function $\psi_1(x)$ and use this to evaluate $|\psi_1(0)|^2/|\psi_1^0(0)|^2$. Plot this exact result along with the approximate result that you obtained in part (d).]

3. A Delta-Function Interaction Between Two Bosons in an Infinite Square Well (5 points)

Do Griffiths problem 6.3.

[The integrals that come up in this problem can certainly all be done by hand; however, there is also nothing wrong with doing them by Mathematica or the equivalent. By this time in your life, you should not spend time doing elementary integrals.]

4. Anharmonic Oscillator (15 points)

Consider the anharmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + \lambda x^3 ,$$

treating the λx^3 term as a perturbation. [Hint: you should not find yourself doing any integrals as you do this problem; you should find yourself manipulating harmonic oscillator creation and annihilation operators and harmonic oscillator energy eigenstates.]

- (a) Show that the first order shift in the ground state energy is zero. Calculate the shift to order λ^2 .

- (b) Calculate the ground state wave function to order λ . (You may just write your answer as a sum of harmonic oscillator states.)
- (c) Sketch the potential $V(x)$ as a function of x for small λ . Is the state you found in (b) anything like the true ground state? What effect has perturbation theory failed to find?
- (d) Now consider instead an anharmonic oscillator with

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + \lambda x^4 ,$$

treating the λx^4 term as a perturbation. Calculate the energy of the ground state to order λ . Sketch $V(x)$ for λ small and positive and for λ small and negative, and comment on what perturbation theory has told you in each case, and in each case comment on whether you think that perturbation theory has given a good approximation to the true ground state energy.

5. Polarizability of a Particle on a Ring; the Ethane Molecule (15 points)

Consider a particle of mass m constrained to move in the xy -plane on a circular ring of radius a . The only variable of the system is the azimuthal angle, which I will call ϕ . The state of the system is described by a wave function $\psi(\phi)$ which must have the property that

$$\psi(\phi + 2\pi) = \psi(\phi)$$

and which must be normalized:

$$\int_0^{2\pi} |\psi(\phi)|^2 d\phi = 1 .$$

- (a) The kinetic energy of the particle can be written:

$$H^0 = \frac{L_z^2}{2ma^2} \tag{2}$$

where $L_z = -i\hbar d/d\phi$. Calculate the eigenvalues and eigenfunctions of H^0 . Which of the energy levels are degenerate?

- (b) Now assume that the particle has a charge q and that it is placed in a uniform electric field ε in the x -direction. We must therefore add to the Hamiltonian the perturbation

$$H' = -q\varepsilon a \cos \phi$$

Calculate the new wave function of the ground state to first order in ε . Use this wave function to evaluate the induced electric dipole moment in the x -direction: $\langle \psi | qx | \psi \rangle$. Determine the proportionality constant between the dipole moment and the applied field ε . This proportionality constant is called the “polarizability” of the system.

- (c) Now consider the ethane molecule CH_3-CH_3 . I am assuming that in some chemistry class you have seen a “balls-on-sticks” description of this molecule. Let us think about how to analyze the rotation of one CH_3 group relative to the other, about the straight line joining the two carbon atoms.

To a zeroth approximation, this rotation is free, and the Hamiltonian H^0 of (2) describes the rotational kinetic energy. (The constant $2ma^2$ must be replaced by some new constant times the moment of inertia of a CH_3 group with respect to the rotational axis. We’ll just keep calling the constant $2ma^2$, though.)

The H' of part (b) is not present in part (c).

We now take the electrostatic interaction energy between the two CH_3 groups into account as a perturbation. We add to H^0 a term of the form

$$H' = b \cos 3\phi ,$$

where b is a real constant. By appealing to your chemistry class balls-on-sticks picture, give a justification for the ϕ dependence of H' . Calculate the energy and wave function of the new ground state (to first order in b for the wave function and to second order for the energy). Give a physical interpretation of the result.