Quantum Physics III (8.06) — Spring 2018

Assignment 7

Posted: Friday, April 13, 2018

Readings

The current reading assignments are:

- Cohen-Tannoudji has a nice treatment of Landau levels, in Ch. VI Complement E.
- Sakurai pp. 130-139.
- Shankar's treatment of Landau levels begins in p. 587.

Notes

In this p-set the cyclotron frequency ω_c and the magnetic length l_B are given by:

$$\omega_c \equiv \frac{qB}{mc}, \qquad l_B \equiv \sqrt{\frac{\hbar}{m\omega_c}} = \sqrt{\frac{\hbar c}{qB}} . \tag{1}$$

The velocity operator $\hat{\mathbf{v}}$ with components $(\hat{v}_x, \hat{v}_y, \hat{v}_z)$ is defined as

$$\hat{\mathbf{v}} = \frac{1}{m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right) . \tag{2}$$

Gauge transformations generated by a function $\Lambda(\vec{x}, t)$ take the form:

$$\begin{aligned} \psi'(\mathbf{x}, t) &\equiv U(\Lambda)\psi(\mathbf{x}, t) \;=\; \exp\left(\frac{iq}{\hbar c}\Lambda(\mathbf{x}, t)\right)\psi(\mathbf{x}, t) \\ \mathbf{A}'(\mathbf{x}, t) &\equiv \mathbf{A}(\mathbf{x}, t) + \vec{\nabla}\Lambda(\mathbf{x}, t), \\ \Phi'(\mathbf{x}, t) &\equiv \Phi(\mathbf{x}, t) - \frac{1}{c}\frac{\partial \Lambda}{\partial t}(\mathbf{x}, t) . \end{aligned} \tag{3}$$

Problem Set 7

1. **Gauge Invariance and the Schrödinger Equation (10 points)**

   (a) Prove the gauge invariance of the Schrödinger equation (SE) by showing that the SE for ψ with the original potentials implies the SE for ψ' with the gauge transformed potentials.

(b) Physical observables are Hermitian operators $\mathcal{O}$ that are gauge *covariant*: Under a gauge transformation with parameter $\Lambda(\mathbf{x}, t)$ taking $\mathcal{O} \to \mathcal{O}'$ we find that

$$\mathcal{O}' = U\mathcal{O}U^{-1},$$

where $U = U(\Lambda)$. Explain why gauge covariant observables have gauge *invariant* expectation values:

$$\langle\psi'|\mathcal{O}'|\psi'\rangle = \langle\psi|\mathcal{O}|\psi\rangle .$$

Answer with brief explanations:

i. Is the sum of gauge covariant operators a gauge covariant operator?
ii. Is the product of gauge covariant operators a gauge covariant operator?
iii. Is $\hat{x}_i$ gauge covariant?
iv. Is $\hat{p}_i$ gauge covariant?
v. Is $\hat{v}_i$ gauge covariant?
vi. Is the Hamiltonian H gauge covariant under arbitrary gauge transformations? If yes, show it. If no, find the class of gauge transformations for which H is gauge covariant.

2. **Classical Motion in a Magnetic Field** (10 points)

Consider a particle of mass m and charge q moving along a trajectory $\mathbf{x}(t)$ through a constant magnetic field along the z-direction $B_z = B$.

(a) Consider motion restricted to the (x, y) plane and use the Lorentz force law

$$m\frac{d\mathbf{v}}{dt} = \frac{q}{c}\mathbf{v} \times \mathbf{B},$$

to show that the general solution for the motion represents circular motion with angular velocity

$$\omega_c = \frac{qB}{mc} . \quad (4)$$

Write your general solution for $x(t)$ and $y(t)$ letting (X, Y) denote the "center of orbit" coordinates, that is, the coordinates of the center of the circle.

(b) Show that X, Y can be expressed in terms of the (time dependent) coordinates (x, y) and the (time dependent) velocities (v_x, v_y) of the particle as

$$X = x + \frac{v_y}{\omega_c}, \qquad Y = y - \frac{v_x}{\omega_c} . \quad (5)$$

Show (by differentiating (5)) that X, Y are constants of motion.

3. **General Aspects of Quantum Motion in a Magnetic Field** (10 points)

The questions in this problem should be derived without explicitly choosing a gauge. Assume the electric field is zero and all motion is in the (x, y) plane.

(a) Consider arbitrary magnetic field (not necessarily constant). Find the commutator $[\hat{v}_x, \hat{v}_y]$ of the velocity operators.

(b) Let the magnetic field be $\mathbf{B} = B\hat{\mathbf{z}}$, with B constant. Motivated by the classical expressions (5) for the center of orbit coordinates, we introduce quantum operators

$$\hat{X} \equiv \hat{x} + \frac{\hat{v}_y}{\omega_c}, \qquad \hat{Y} \equiv \hat{y} - \frac{\hat{v}_x}{\omega_c}. \tag{6}$$

Are $\hat{X}$ and $\hat{Y}$ gauge covariant? Find the commutator $[\hat{X}, \hat{Y}]$. The $\hat{X}$ and $\hat{Y}$ operators are the simplest example in physics of **non-commutative** coordinates!

(c) Show that the coordinates $\hat{X}$ and $\hat{Y}$ are conserved:

$$[\hat{X}, H] = [\hat{Y}, H] = 0 . \tag{7}$$

This, together with the non-commutation of $\hat{X}$ and $\hat{Y}$ imply that at most one of $\hat{X}$ and $\hat{Y}$ can be diagonalized together with the Hamiltonian. [Hint: It is convenient to write the Hamiltonian in a form $H = \frac{1}{2}m(\hat{v}_x^2 + \hat{v}_y^2)$ and first find the commutators between $\hat{X}, \hat{Y}$ and $\hat{v}_x, \hat{v}_y$.]

(d) Define the operator $\hat{R}^2$ as the distance square of the orbit center to the origin

$$\hat{R}^2 \equiv \hat{X}^2 + \hat{Y}^2 .$$

Find the spectrum of the operator $\hat{R}^2$. [Hint: Think of $\hat{R}^2$ as a harmonic oscillator Hamiltonian.]

(e) Define the *orbit radius* operator r_c via the classically inspired relation

$$r_c^2 \equiv (\hat{x} - \hat{X})^2 + (\hat{y} - \hat{Y})^2 . \tag{8}$$

Find the spectrum of r_c^2. [Hint: Write r_c^2 in terms of velocities].

(f) The angular momentum operator $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ is not gauge covariant. To define a gauge invariant version $\hat{\mathcal{L}}_z$ we take

$$\hat{\mathcal{L}}_z \;=\; \hat{x}\, m\hat{v}_y - \hat{y}\, m\hat{v}_x + \cdots \tag{9}$$

where the dots are terms that you should determine in terms of B, $\hat{x}$, $\hat{y}$ and other constants, using the condition that $\hat{\mathcal{L}}_z$ reduces to the familiar $\hat{L}_z$ in the circular gauge $(A_x, A_y) = \frac{1}{2}B(-y, x)$.

One way to see that this "angular momentum" $\hat{\mathcal{L}}_z$ is a constant of the motion is to show it is related to other constants of the motion. Show that $\hat{\mathcal{L}}_z$ is proportional to $\hat{R}^2 - r_c^2$.

4. **Electromagnetic Current Density in Quantum Mechanics** (10 points)

The probability flux in the Schrödinger equation can be identified as the electromagnetic current density, provided the proper attention is paid to the effects of the vector potential. Without electromagnetic fields the probability current $\mathbf{J}$ is

$$\mathbf{J}(\mathbf{x}, t) = \frac{\hbar}{m}\text{Im}\,[\psi^* \nabla \psi] \;=\; \frac{1}{m}\text{Re}\Big[\psi^* \,\frac{\hbar}{i}\nabla\, \psi\Big] = \frac{1}{m}\text{Re}\Big[\psi^*\,(\hat{\mathbf{p}}\,\psi)\Big]\,, \tag{10}$$

using $\text{Im}(z) = \text{Re}(z/i)$ and noting that $\hat{\mathbf{p}}$ is acting on the wavefunction to the right. The probability current $\mathbf{J}(\mathbf{x}, t)$ is not an operator, it is a function of position and time. In the presence of electric and magnetic fields, the probability current is modified to

$$\mathbf{J}(\mathbf{x}, t) = \frac{\hbar}{m}\text{Im}\left[\psi^* \nabla \psi\right] - \frac{q}{mc}\psi^* \psi \mathbf{A} \;=\; \frac{1}{m}\text{Re}\Big[\psi^* \Big(\frac{\hbar}{i}\nabla - \frac{q}{c}\mathbf{A}\Big)\psi\Big] = \text{Re}\left(\psi^* \hat{\mathbf{v}}\, \psi\right) . \quad (11)$$

This probability flux is conserved and when multiplied by the particle charge q it can be interpreted as the electromagnetic current density $\mathbf{j} \equiv q\mathbf{J}$.

(a) Derive the expression eq. (11) for the probability flux. [Hint: The derivation of eq. (11) is parallel to that of (10), i.e. you begin with the conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0\,,$$

use $\rho = \psi^* \psi$ and determine the current $\mathbf{J}$ that makes it work.]

(b) Show that $\mathbf{j} = q\mathbf{J}$ has units of electric current density.

Show that $\mathbf{J}(\mathbf{x}, t)$ is a gauge *invariant* function of $\mathbf{x}$ and t: $\mathbf{J}'$ calculated in terms of $\mathbf{A}'$ and ψ' is identical to $\mathbf{J}$ calculated in terms of $\mathbf{A}$ and ψ.

5. **The Aharonov-Bohm Effect on Energy Eigenvalues**(10 points)

The Aharonov-Bohm effect modifies the energy eigenvalues of suitably chosen quantum mechanical systems. In this problem, we work out the example that Griffiths discusses in 385-387.

Imagine a particle constrained to move on a circle of radius b (a bead on a wire ring, if you like.) Along the axis of the circle runs a solenoid of radius $a < b$, carrying a magnetic field $\mathbf{B} = (0, 0, B_0)$. The field inside the solenoid is uniform and the field outside the solenoid is zero. The setup is depicted in Griffiths' Fig. 10.10.

(a) Construct a vector potential $\mathbf{A}$ which describes the magnetic field (both inside and outside the solenoid) and which has the form $A_r = A_z = 0$ and $A_\phi = \alpha(r)$ for some function $\alpha(r)$. We are using cylindrical coordinates z, r, ϕ.

(b) Now consider the motion of a "bead on a ring": write the Schrödinger equation for the particle constrained to move on the circle $r = b$, using the $\mathbf{A}$ you found in (a). [Hint: the answer is given in Griffiths.]

(c) Solve the Schrodinger equation and find the energy eigenvalues and eigenstates.

(d) Plot the energy eigenvalues as a function of the enclosed flux Φ. Show that the energy eigenvalues are periodic functions of Φ with period Φ_0, where you must determine Φ_0. For what values of Φ does the enclosed magnetic field have no effect on the spectrum of a particle on a ring? Show that the Aharonov-Bohm effect can only be used to determine the fractional part of Φ/Φ_0.

(e) Suppose we introduce a defect on the ring at $\phi = 0$, which can trap the particle, i.e. in addition to the states you worked out above, there now exist trapped states in which the wave function of the particle is localized around $\phi = 0$. For simplicity, assume the trapped state wave functions vanish outside an interval $(-\phi_0, \phi_0)$ for some $\phi_0 < \pi$. Show that the energy of a trapped state does NOT depend on the existence of the solenoid.

[Hint: Find a gauge in which the vector potential vanishes identically in the region where the trapped state wavefunctions are supported. You should also explain why the same argument does not apply to states of part (c).]

[Moral of problem: even though the bead on a ring is in a region in which $\mathbf{B} = 0$, the presence of a nonzero $\mathbf{A}$ affects the energy eigenvalues of states whose wave functions cover the whole circle. The vector potential does *not* affect the energies of localized states. This is the counterpart for the energy spectrum of the statement that the Bohm-Aharonov interference pattern is shifted *if and only if* the relevant paths enclose the solenoid.]

6. **Adiabatic Spin Rotation (15 points)**

Consider a spin one-half particle at rest, with its spin free to rotate in response to a time-dependent magnetic field. The Hamiltonian of the system is

$$H = -\frac{2\mu_0}{\hbar}\,\mathbf{S}\cdot\mathbf{B}(t)\ . \tag{1}$$

Assume $\mu_0 > 0$. We will start at time $t = -T$ with a large magnetic field mostly in the $-\hat{z}$ direction which we will slowly decrease to zero and then increase in the opposite direction until time $t = T$. At the same time, we will assume that there is a constant small field in the (x, y) plane. The magnetic field is then

$$\mathbf{B}(t) = (B_x, B_y, \gamma t) \qquad \text{for } -T \le t \le T. \tag{2}$$

Here $\gamma > 0$ is the rate at which the magnetic field in the z direction grows and we assume that

$$\gamma T \gg \sqrt{B_x^2 + B_y^2}\,, \tag{3}$$

so that for $t = \pm T$ the magnetic field is mostly along the z axis. Denote the ground state at time t by $|\psi_+(t)\rangle$ and the excited state at time t by $|\psi_-(t)\rangle$. These correspond, respectively, to spins that are either aligned or anti-aligned with the magnetic field. Suppose that at time $t = -T$, the spin is in state $|\psi_+(-T)\rangle$.

(a) Use the adiabatic theorem to argue that the particle initially in the state $|\psi_+(-T)\rangle$ finishes in the state $|\psi_+(T)\rangle$ (up to an overall phase) with nearly unit probability, as long as $|\delta B| \equiv \sqrt{B_x^2 + B_y^2} \neq 0$. Find the inequality satisfied by γ needed for the adiabatic theorem to apply.

(b) Explain why condition (3) implies $|\psi_+(-T)\rangle \approx |-\rangle$ and $|\psi_+(T)\rangle \approx |+\rangle$. In this case the adiabatic process in (a) will convert the state from $\approx |-\rangle$ to $\approx |+\rangle$. Use condition (3) and the adiabatic condition you found in part (a) to find $T \gg \ldots$, where the quantity in dots does not depend on γ.

(c) Instead of a time-varying magnetic field, consider a *spatially* varying magnetic field. We can use the adiabatic theorem to understand the magnetic traps used by MIT atomic physicists to trap very cold gases of *spin-polarized* atoms. Classically, the force on a dipole $\boldsymbol{\mu}$ from a magnetic field gradient is

$$\mathbf{F} = (\boldsymbol{\mu} \cdot \nabla)\mathbf{B}(\mathbf{x}). \tag{4}$$

One can design magnetic field gradients such that an atom which has its spin antiparallel to the local magnetic field $\mathbf{B}(\mathbf{x})$ experiences a force toward the center of the trap. Those atoms with spins parallel to $\mathbf{B}(\mathbf{x})$ feel a force which expels them from the trap. This way, one can trap only atoms of one polarization state.

But this raises a question: Since $\mathbf{B}(\mathbf{x})$ varies in space, how can we ensure that the thermal movements of the atoms within the trap preserve the property that their spins are always anti-aligned with the local magnetic field?

Assume that the atoms have mass m and temperature T. Let B denote the magnitude of the magnetic field and $|\nabla B|$ denote the magnitude of its gradient. State an inequality in terms of $m, k_B, \hbar, T, \mu_0, B, |\nabla B|$ that must be satisfied for the adiabatic condition to be valid. [Hint: Since B varies in space, the thermal motion of the atoms implies that the spins experience time-dependent magnetic fields.]

(d) Since it is magnetic *gradients* which exert the trapping forces, you might think that there would be no problem if at one point in the trap $\mathbf{B} = 0$. In fact, this *does* cause problems, and the traps are designed to have $\mathbf{B} \neq 0$ everywhere. Use your result in (c) to explain why.

7. **Landau-Zener transitions (15 points)**

Consider the Hamiltonian

$$H = \begin{pmatrix} E_1 & H_{12} \\ H_{12}^* & E_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\alpha t & H_{12} \\ H_{12}^* & -\frac{1}{2}\alpha t \end{pmatrix}, \qquad \alpha > 0\,,$$

with α and H_{12} time-independent constants, and the "diabatic" basis

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We have the natural ansatz for a time-dependent solution

$$|\Psi, t\rangle = A(t) e^{-\frac{i}{\hbar}\int_0^t E_1(t')dt'} |1\rangle + B(t) e^{-\frac{i}{\hbar}\int_0^t E_2(t')dt'} |2\rangle\,.$$

Here $A(t)$ and $B(t)$ are functions of time to be determined.

(a) Use the Schrödinger equation to derive a set of linear, first-order, coupled differential equations for $A(t)$ and $B(t)$. You can leave you answer in terms of H_{12}, H_{12}^* and $E_{12}(t) \equiv E_1(t) - E_2(t)$. Derive a second-order linear differential equation for $B(t)$.

(b) Assume that at $t = -\infty$, the system is in state $|2\rangle$, so we have

$$B(-\infty) = 1\,. \qquad A(-\infty) = 0.$$

Verify that constant A and constant B is a solution for $H_{12} = 0$. This is a zeroth-order solution. Consider now H_{12} as a very small perturbation. Find the first-order in H_{12} equation for $A(t)$ and determine $A(t = \infty)$ in terms of $H_{12}, \alpha, \hbar$ and numerical constants. To this approximation, what is the probability P that the system remains in $|2\rangle$ at $t = \infty$? (Hint: in calculating $A(t = \infty)$ you will encounter the *Fresnel* integrals).

(c) Reconsider the second-order differential equation for $B(t)$ derived in part (a). Construct a unit-free time parameter τ in terms of t, α, and $|H_{12}|$ (and no extra numerical constants) and show that the differential equation for $B(\tau)$ becomes

$$\frac{d^2B}{d\tau^2} + i\gamma\,\tau\frac{dB}{d\tau} + \gamma^2 B \;=\; 0\,, \qquad \gamma = \frac{|H_{12}|^2}{\hbar\alpha}\,.$$

Solve numerically this equation for $\gamma = 0.1$, and integrating your solution in the interval $\tau \in (-25, 25)$. Plot the value of $|B(\tau)|^2$ which represents the probability for the system to have made a non-adiabatic transition by time τ. Compare the value of $|B(\infty)|^2$ that you can glean from your graph, with the prediction that the probability for a non-adiabatic transition is $\exp(-2\pi\gamma)$.