PROFESSOR: Let us consider the anharmonic oscillator, which means that you're taking the unperturbed Hamiltonian to be the harmonic oscillator. And now, you want to add an extra term that will make this anharmonic. Anharmonic reflects the fact that the perturbations are oscillations of the system are not exactly harmonic. And in the harmonic oscillator, the energy difference between levels is always the same. That's a beautiful property of the harmonic oscillator.

That stops happening in an anharmonic oscillator. The energy differences can vary. So the things, if you have a transition from one level, first level to the ground state, or second level to the ground state, one is not the harmonic of the other because they're not exactly twice as big as each other.

So let's try to add an x to the 4th perturbation, which is intuitively very clear. You have a potential. And you're adding now an extra piece that behaves like x to the 4th. And it's going to make this potential blow up faster.

Now, in order to get the units right, we need the length scale. There's a length scale d in the harmonic oscillator. We're going to use it.

One way to derive that length scale is to recall from units that p, a momentum, has units of h divided by length. So b squared has units of h squared over d squared. There's an m. So that's an energy.

And this is an energy too. So we can set it equal to m omega squared d squared. From where we get, for example, as 1 over d to the 4th is equal to m squared omega squared over h squared. And d squared is equal to h over m omega.

So that's a length scale, something with units of length in the harmonic oscillator. So if we want to add the perturbation that is x to the 4th, we'll add a lambda delta H that is going to be a lambda, which is unit freedom, something that has units of energy. So you can put h bar mega has units of energy.

And then you can put the operator x to the 4th divided by d to the 4th. It's a perturbation with units of energy. This is good.

So there's a couple of ways of thinking of it. You may remember that in the harmonic oscillator x, the operator x, was given by the square root of h over 2m omega, a plus a dagger. So this is d times a plus a dagger over square root of 2.

So this perturbation lambda delta H is equal to lambda. H bar omega would be here. H bar omega. Now x/d is this a plus a dagger divided by square root of 2. So you have a 4a plus a dagger to the 4th. So that's lambda delta H. That's your perturbation.

So what do we want to do with this perturbation? It's a nice perturbation. We want to compute the corrections to the energy, corrections to the states, and see how they go. We can do that easily.

Let's compute first that corrections to the ground state energy. These are calculations that are not difficult, but they take some care. As any calculation, it's based on just getting a lot of numbers right.

So what is the ground state energy correction? So we had a series of formulas that I just erased. But this index k0 for our state, we can think of k as being the eigenvalue of the number operator. So we'll get n.

And remember in the harmonic oscillator, we label states by 0, 1, 2, 3. And that's the eigenvalue of the number operator in the harmonic oscillator. So the ground state energy would correspond to 0, has a first order correction that would be given by the unperturbed ground state and delta H 0.

So what is that? Delta H is h omega over 4. And we have 0 a plus a dagger to the 4th 0. Easy enough. That's what we have to do.

Now, the evaluation of that matrix element is a simple exercise, the kind of things you've been doing before. You have to expand and just use the aa dagger computation relation that this aa dagger is equal to 1.

You said repeatedly a kills the vacuum. a dagger kills the other vacuum on the left. Please do it. If you feel you're out of practice, the answer is the number 3 here. So e01 is equal to 3/4 h bar omega.

So how does that tell you anything? Well, better use the full formula. So e0 of lambda is supposed to be equal to the ground state energy unperturbed, which was h omega over 2 plus lambda times the first order correction, which is 3/4 h bar omega plus order lambda squared.

So e0 of lambda is h omega over 2, 1 plus 3/2 lambda plus order lambda squared. That's what we got. And that makes nice sense. Your ground state energy you knew used to be h omega over 2. If you introduce this perturbation, you don't expect it to jump abruptly. It's going to grow up with the lambda to the 4th term with the x to the 4th term into 3/2 lambda here, a small correction.

So far, so good. What takes a bit more work is doing a higher order correction. So let's do the next one as well.

So for this second order correction, we have a formula. And let's see what it tells us. It tells us that the second order correction to the energy is minus the sum over all states that are not the ground states. So it's k different from 0. That's good enough. Delta H 0k squared over Ek0 minus e00.

So potentially, it's an infinite sum. And that's the kind of thing that is sometimes difficult. In many examples, you have infinite sums. Sometimes you can do an approximation, say a few terms is all you need to do. But in this case, happily, it's not going to be an infinite sum.

So what do we have? Delta H0k is what we need to calculate. The energy differences, we know. These are the unperturbed harmonic oscillator energy differences.

What we need is this object, which is h bar omega over 4. Remember, what is delta H there in that box formula there. And we have 0 a plus a dagger to the 4th K, where K is a state with number K.

If you wish, you could say K is a dagger to the K over square root of K factorial acting on the ground state. But sometimes you don't need that. You actually don't need to put the value of K.

So our challenge here is to see which values of K exist. For that, you can think of this part of the term, the a plus a dagger to the 4th acting on the vacuum. This is like, in a sense, this term is nothing else but in saying what happens to the wave function when you multiply it by x to the 4, to the ground state wave function.

That's physically what you're doing here. You're multiplying an x to the 4th times the ground state wave function. So it should give you an even wave function. So you would expect that x to the 4th multiplied by 5 0 should be proportional to 5 0, but maybe 5 2 and 5 4th.

Should not have the odd ones because this function is not odd. It's even. So as you express x to the 4th 5 0 in terms of the other ones, it should be that. So indeed, again, I ask you to do a little computation. If you do 0 a plus a dagger to the 4th, this is equal to-- you can show. It gives you 4 factorial square root the state 4 plus 6 square root of 2 times the state 2 plus 3 times times the ground state.

OK. So we have that look let's continue therefore. Given that [INAUDIBLE] has given us three states-- one proportional to the vacuum, one proportional to the state with occupation number 2, and one with a state of occupation number four-- the values of k that are relevant, since k is different from 0, are only 2 and 4. 0 doesn't matter. And only 2 and 4 can couple here.

So what do we get? Delta H0 4 would be, from this formula, h omega over 4 times this inner product when k is equal to 4. So you get here square root of 4 factorial, which is h omega over 2 times square root of 6.

Delta H02 is equal to h omega over 4. And the overlap of this state with 2, which is 6 square root of 2. So this is h omega 3 square root of 2/2.

OK. So that's your first step. When you have to compute a perturbation, you have to compute all the relevant matrix elements. And you have to think what you can get. And only k equal 2 and 4 happen, so this is the sum of two terms.

So the second order correction to the energy is minus delta H0 2 squared divided by E2 0 minus E0 0 minus delta H0 4 squared over E4 0 minus E0 0.

So what is this? It's minus delta H0 2 squared is h bar omega squared. 9 times 2/4. 9 times 2/4. And you divide by this difference. And the difference between E2 and E0 is 2 h bar omega. There's an h bar omega difference every state. So you go from E0 to E1 to E2, two of them.

Minus delta H0 4 squared, which is H omega squared times 6/4 over 4 h bar omega. Well, this gives you minus 21/8 h bar omega.

And therefore, the energy, this is the E0 2, so you can go back here and write E0 lambda is equal to H omega over 2. 1 plus 3/2 lambda. And then the next term that we now calculated is 21/4 lambda squared.

Very good. We work hard. We have lambda squared correction to this energy.

Now, this is not really trivial. There's no analytic way to solve this problem. Nobody knows an analytic way to solve this problem.

So this is a nice result. It's so famous the problem that people have written hundreds of papers on this. The first people that did a very detailed analysis was Bender and Wu. And they computed probably 100 terms in the series or so.

The next term turns out to be 333/8 lambda cube. Followed by 30,855/64 lambda to the 4th plus ordered lambda to 5th. Funnily, these people also proved that this series has no radius of convergence. It never converges. It just doesn't converge at all.

You would say, oh, what a waste of time. No, it's an asymptotic expansion, this series. So it can be used.

So it's a very funny thing. Series sometimes don't converge so well. And this one is a peculiarly non-convergent series.

So what does it mean only that it's an asymptotic expansion? Basically, the expansion for the factorial that you use in statistical mechanics all the time, Sterling's expansion, is an asymptotic expansion. Quantum electro dynamics with Feynman rules is an asymptotic expansion.

Very few things actually converge in physics. That's the nature of things. What it means is that when you take lambda, for example, to be 1 in 1,000, a very small number, then this is 1. This will be very small. This will still be small. This will be small. And eventually, the terms go smaller, smaller, smaller, smaller, smaller, smaller. And eventually, start going up again.

And if you have an asymptotic expansion, you're supposed to stop adding terms until they become the smallest. And that will be a good approximation to your function. So this can be used. You can use it. You can do numerical work with the states and compare with asymptotic expansion. And it's a very nice thing.

Another thing is that you can compute the first correction to the state. I will leave that as an exercise. And in the notes, you will find the expression of the first correction to this state. It's not much work because, after all, it involves just this kind of matrix elements. So it's basically using the matrix elements to write something else.