Massachusetts Institute of Technology

Department of Physics

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Problem Set 2

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Problem 1: Stellar Aberration and Parallax [20 pts]

Suppose I have a star located at a distance R from the center of the solar system which makes an angle θ with respect to the Earth orbital plane (ecliptic). You observe the star from Earth at various times throughout the year. Take the Earth-Sun distance as r_e and the angular velocity of the Earth's orbit as ω, such that the orbital velocity is $v = \omega r_e$. In this problem, $R \gg r_e$ and $c \gg v$.

(a) Find the maximum and minimum change in the observed angle of the star in the sky due to the change in the Earth's position over any six-month period (parallax).
[*Hint*: For this problem, work in a coordinate system with the Sun at the origin, and the Earth's orbit in the xy plane. Calculate the time evolution of the Earth-star vector (vector pointing from the Earth to the star), and derive the change in the star's observed angle from here. To better visualize changes in the observed angle, consider cases where there are only changes in the elevation and azimuthal angles first.]

(b) Perform the same analysis except look at the effect due to the Earth's velocity (aberration). For this problem, you may assume that the starlight coming to the Earth is parallel to that coming to the Sun.
[*Hint*: Make an analogy between this problem and part(a). Specifically, think about the vectors of interest for this problem.]

(c) Which effect would dominate for most observations? Take as an example one of our closest stars, Alpha Centauri, about 4.4 light years away.

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(a) Let us work in the frame where the Sun is the center of our coordinate system. Let us define a vector that points from the sun to the observed star and call that vector $\vec{R}$. If we write out the coordinates of R, we find

$$\vec{R} = R(\cos\theta, 0, \sin\theta)$$

Notice that, without any loss of generality, I have aligned the x-axis such that it aligns with the projection of the vector pointing from the sun to the star in the Earth's orbit plane. Next, I define the location of Earth within this coordinate system, calling that vector $\vec{r}_e(t)$.

$$\vec{r}_e(t) = r_e(\cos(\omega t), \sin(\omega t), 0)$$

The vector that describes the position from Earth to the star is $\vec{s}(t) = \vec{R} - \vec{r}_e(t)$. Both the direction and the length of the vector has changed as a result of the shift in the Earth's orbit. The quantity we are after is the (normalized) vector $\vec{s}$, which is given by...

$$\hat{s}(t) = \frac{\vec{R} - \vec{r}_e(t)}{|\vec{R} - \vec{r}_e(t)|}$$
$$\hat{s}(t) = \frac{\vec{R} - \vec{r}_e(t)}{\sqrt{R^2 + r_e^2(t) - 2\vec{R}\cdot\vec{r}_e(t)}}$$
$$\hat{s}(t) = \frac{\vec{R} - \vec{r}_e(t)}{\sqrt{R^2 + r_e^2 - 2Rr_e\cos\theta\cos(\omega t)}}$$

To simplify things a bit, let us define η as r_e/R and let us expand the bottom expression in terms of η.

$$\hat{s}(t) = \frac{\vec{R} - \vec{r}_e(t)}{R\sqrt{1 + \eta^2 - 2\eta\cos\theta\cos(\omega t)}}$$
$$\hat{s}(t) = \left(\frac{\vec{R} - \vec{r}_e(t)}{R}\right)(1 + \eta\cos\theta\cos(\omega t) + \mathcal{O}(\eta^2))$$

Writing out all components and keeping only terms of order η, we find...

$$\hat{s}(t) = (\cos\theta - \eta\cos(\omega t), -\eta\sin(\omega t), \sin\theta)(1 + \eta\cos\theta\cos(\omega t) + \mathcal{O}(\eta^2))$$
$$\hat{s}(t) = (\cos\theta + \eta(\cos^2\theta\cos(\omega t) - \cos(\omega t)), -\eta\sin(\omega t), \sin\theta + \eta\sin\theta\cos\theta\cos(\omega t)) + \mathcal{O}(\eta^2)$$
$$\hat{s}(t) = \hat{R} + \eta(\cos^2\theta\cos(\omega t) - \cos(\omega t), -\sin(\omega t), \sin\theta\cos\theta\cos(\omega t))$$

We are asked to look at the change in angle over six months of observations, which is equivalent to shifting the time variable by π/ω.

$$\hat{s}(t + \pi/\omega) - \hat{s}(t) = -2\eta(\cos^2\theta\cos(\omega t) - \cos(\omega t), -\sin(\omega t), \sin\theta\cos\theta\cos(\omega t))$$

Now, let us look at the maxima and minima of this equation. Suppose we let $t = \pi/(2\omega)$? Then, the change is observed only along the y-axis, and we have...

$$\hat{s}(3\pi/(2\omega)) - \hat{s}(\pi/(2\omega)) = (0, 2\eta, 0)$$

That angle *used* to be zero, but now is has changed by a total of 2η. To find the angular change, Let α be the change in angle due to parallax along the y-axis. That means...

$$\sin(\alpha) - \sin(-\alpha) = 2\eta$$
$$\boxed{\alpha = \arcsin\eta \simeq \eta}$$

What about at t=0 (the other extreme)? There, the change is mainly in the x-z plane (i.e. the plane defined by the Earth-star).

$$\hat{s}(\pi/\omega) - \hat{s}(0) = -2\eta(\cos^2\theta - 1, 0, \sin\theta\cos\theta)$$
$$\hat{s}(\pi/\omega) - \hat{s}(0) = -2\eta\sin\theta\ (-\sin\theta, 0, \cos\theta)$$

What kind of angle change does this correspond to? It is really a tweak to the original position of the star. So let us come back to the original vector $\hat{R}$ and add a small angle δ to θ. After 6 months, the sign of δ will change, so we have...

$$\hat{R}'(\pi/\omega) = (\cos(\theta + \delta), 0, \sin(\theta + \delta))$$
$$\hat{R}'(0) = (\cos(\theta - \delta), 0, \sin(\theta - \delta))$$
$$\hat{R}'(\pi/\omega) - \hat{R}'(0) = 2\sin\delta(-\sin\theta, 0, \cos\theta)$$

On inspection, one can see the two equations match, allowing one to relate δ to η

$$2 \sin \delta = -2\eta \sin \theta$$
$$\boxed{\delta = \arcsin\left(-\eta \sin \theta\right)}$$

(b) For the shift in position in the sky, we need to look at the velocity of the Earth-star system instead. To find the Earth's velocity as a function of time, we take the time derivative of $\vec{r}_e$.

$$\vec{v}(t) = \frac{d\vec{r}_e(t)}{dt} = \omega r_e(-\sin(\omega t), \cos(\omega t), 0)$$
$$\vec{v}(t) = v(-\sin(\omega t), \cos(\omega t), 0)$$

where $v = \omega r_e$. As expected, the velocity vector is tangental to the radius of the Earth's orbit. What about the velocity of the incoming light? That is coming to us along the same direction as our previous R, except it is toward us rather than toward the star. In other words...

$$\vec{c} = -c(\cos\phi, 0, \sin\phi)$$

Technically, the angle ϕ is not exactly the angle with respect to the sun (previously θ), but the angle of the Earth with respect to the star (aberration angle). But since the parallax effect is small enough to be negligible, I can treat ϕ and θ as essentially the same.

Now, I am going to cheat a little bit here. Instead of going through the entire derivation as I did in part (a), I realize that the problems are essentially identical, for two adjustments: (a) the vector pointing to the star reverses sign and (b) the velocity vector is a quarter period ($\pi/(2\omega)$) out of phase with the Earth's position vector. That means, if we had a maxima between Jan-June, now it is during September-March (you may remember this from your reading in French). I can therefore substitute η for β where $\beta = v/c$. Then, I find...

$$\hat{s}(\pi/\omega) - \hat{s}(0) = -2\beta \sin\phi \ (-\sin\phi, 0, \cos\phi)$$
$$\boxed{\delta = \arcsin\left(-\beta \sin \phi\right)}$$

and

$$\hat{s}(3\pi/(2\omega)) - \hat{s}(\pi/(2\omega)) = (0, 2\beta, 0)$$
$$\boxed{\alpha = \arcsin\left(\beta\right)}$$

(c) As we found in Problem Set 1, β for the Earth's orbital velocity is $\simeq 10^{-4}$. What about η? Well, 4.4 light years is approximately 4.2×10^{16} meters, while the earth (at 1 AU) is about 1.5×10^{11} meters from the Sun. That makes η about 3×10^{-6}, or about 30 times smaller than the aberration effect (explains why aberration was discovered first).

Problem 2: Binary Stars (from Resnik, Ch 1) [20 pts]

Consider one star in a binary system moving in a uniform circular motion with speed v. Consider two positions: (I) the star is moving *away* from the Earth along a line connecting them, and (II) the star is moving *toward* the Earth along the line connecting them. Let the star's period of motion be T and it's distance be L. Assume that L is large enough that positions (I) and (II) are a half-orbit apart. Assume that you will be testing a set of *emission theories*, whereby the motion of the source is imparted to the velocity of the emission.

1. Find the time it would take for the star to appear to move from position I to position II, and from position II to position I.
2. Show that the star would appear *both* at position I and II if $T/2 = \frac{2lv}{c^2-v^2}$.

- In emission theories, the premise was that the velocity of light depended on the speed of the source (in direct contrast with Maxwell's theory). But binary systems could be a means to test this scenario.

Imagine that at $t = 0$ the star in question is at position I and emits a pulse of light toward earth. Because the star is receding from us, the velocity of light (according to this theory) is $c - v$. The time it takes to get to earth (call that t_1 is given by)

$$t_1 = \frac{l}{c-v}$$

After half a period ($T/2$), the star now sends a second pulse of light also toward Earth, but this time the star is approaching Earth. That pulse of light will arrive at a time t_2 defined as

$$t_2 = \frac{l}{c+v} + \frac{T}{2}$$

On Earth, we would compute the half-period based on the time difference $t_2 - t_1$, which yields...

$$T_E(I \to II) = \frac{l}{c+v} + \frac{T}{2} - \frac{l}{c-v}$$
$$T_E(I \to II) = \frac{T}{2} - \frac{2lv}{c^2-v^2}$$

By symmetry, if one goes from position II (toward Earth) to position I (away from Earth), it is equivalent to flipping the sign on v, hence

$$T_E(II \rightarrow I) = \frac{T}{2} + \frac{2lv}{c^2 - v^2}$$

For part (b), note that if $T/2 = \frac{2lv}{c^2-v^2}$, then $T_E(I \rightarrow II) = 0$ while $T_E(II \rightarrow I) = \frac{4Lv}{c^2-v^2}$, which is just T (a full 360 deg rotation). So, the star appears to be at the same position. Note that this is true for any multiple of the period as well.

Problem 3: Interferometers [20 pts]

Consider a laser interferometer which emits a monochromatic beam of wavelength λ, arranged as in a Michelson-Morley configuration with two perpendicular arms. The length of each arm is given by $l_1 = l$ and $l_2 = l + \delta l$. We wish to attempt to measure the velocity of the apparatus with respect to an ether "wind" which is arbitrarily pointed in some direction with respect to the first arm (call that angle ϕ). Throughout this problem, keep your answer only to the first non-vanishing order of β.

(a) Assume for the moment that the two arm lengths are the same (i.e. $\delta l \rightarrow 0$). Compute the change in the number of fringes detected under this more general case where the velocity is not aligned with the detector. Remember that one rotates a interferometer with perpendicular arms by 90° to check for changes in the fringe pattern.

(b) Now take the more general case where the two lengths are not equal. What is the change in the number of fringes with an interferometer of unequal arm lengths?

(c) Show that in the limit $\delta l \rightarrow 0$ and the velocity is aligned, you reproduce the MM result.

(d) Suppose you simply wanted to show whether the ether effect existed (i.e. to detect an change in the number of fringes). Would it matter if the lengths were not exactly equal? What about if the apparatus was properly aligned? What would you need to be careful in order not to fool yourself you measured a positive result? [Note: No equations here. Just for you to think about.]

- Suppose I have one arm of the MM interferometer of length l making an angle ϕ with respect to the ether wind. The amount of time t it takes to go from one end to the other can be found by looking at the distance traveled by the light. Suppose the arm is aligned along the x-axis. In this case, there will be a component of the velocity along the x-axis ($v \cos \phi$) and one perpendicular to it ($v \sin \phi$). To solve for t, one solves the quadratic equation...

$$(l + vt\cos\phi)^2 + (vt\sin\phi)^2 = c^2t^2$$
$$t = (\frac{l}{c})\frac{\beta\cos\phi + \sqrt{1-\beta^2\sin^2\phi}}{1-\beta^2}$$

The latter shows only the positive time solution, which is what we are interested in. On the return trip, the angle between the spectrometer arm and the velocity flip direction ($\phi \to \phi + \pi$). Adding the two trip times, we find...

$$t \equiv t(l,\phi) = (\frac{2l}{c})\frac{\sqrt{1-\beta^2\sin^2\phi}}{1-\beta^2}$$

This function is actually very useful now because it generalized the propagation time as a function of distance traversed (l) and angle with respect to the ether (ϕ). We can use this formula to help us find the answers to various parts.

(a) We are told that $\delta l = 0$ (i.e $l_2 = l_1 = l$). Since the spectrometer has arms 90 degrees apart, we wish to compare the travel time between the two arms, Δt

$$\Delta t_1 = t(l, \phi_1 - \pi/2) - t(l, \phi_1)$$
$$\Delta t_1 = \frac{2l}{c}\frac{(\sqrt{1-\beta^2\cos^2\phi_1} - \sqrt{1-\beta^2\sin^2\phi_1})}{1-\beta^2}$$

Expanding to second order, we find...

$$\Delta t_1 = \frac{2l}{c}(\frac{\beta^2}{2}(\cos^2\phi_1 - \sin^2\phi_1))(1+\beta^2) + \mathcal{O}(\beta^4)$$
$$\Delta t_1 = \frac{2l}{c}(\frac{\beta^2}{2}\cos 2\phi_1)(1+\beta^2) + \mathcal{O}(\beta^4)$$
$$\Delta t_1 = \frac{l\beta^2}{c}(\cos 2\phi_1)$$

Now, suppose that I take the measurement by rotating the whole apparatus at a different angle ϕ_2, to provide a new time difference Δt_2. If I look at the difference of differences, I find...

$$\Delta t_2 = \frac{l\beta^2}{c}(\cos 2\phi_2)$$
$$\Delta t_1 - \Delta t_2 = \frac{l\beta^2}{c}(\cos 2\phi_1 - \cos 2\phi_2)$$

In the Michelson-Morley experiment, we interchange the two arms, which is equivalent to letting $\phi_2 = \pi/2 - \phi_1$. That means $\cos 2\phi_2 = -\cos 2\phi_1$ and we find...

$$\Delta t_1 - \Delta t_2 = \frac{2l\beta^2}{c}\cos 2\phi_1$$

Finally, converting this to a change in fringe pattern (i.e. $\delta = \frac{c(\Delta t_1 - \Delta t_2)}{\lambda}$), we have...

$$\boxed{\delta = \frac{2l\beta^2}{\lambda}\cos 2\phi_1}$$

People may simply keep things in terms of ϕ_2 and ϕ_1 (perfectly acceptable). In which case, the answer is more generally...

$$\boxed{\delta = \frac{l\beta^2}{\lambda}(\cos 2\phi_1 - \cos 2\phi_2)}$$

(b) Next we are told to consider the case where l_2 is not the same as l_1, i.e. $l_2 = l + \delta l$. I can still use by general path through one arm of the spectrometer in order to calculate the time of propagation. I can also note (which makes my life a lot simpler) that the time of travel is linear in the length parameter. That is,

$$t(l + \delta l, \phi) = t(l, \phi) + t(\delta l, \phi)$$

So, in calculating the optical path difference between the two arms, $\Delta t_1'$, things become slightly easier.

$$\Delta t_1' = t(l + \delta l, \phi_1 - \pi/2) - t(l, \phi_1)$$
$$\Delta t_1' = t(l, \phi_1 - \pi/2) - t(l, \phi_1) + t(\delta l, \phi_1)$$
$$\Delta t_1' = \Delta t_1 + t(\delta l, \phi_1)$$

Likewise for $\Delta t_2'$, we have...

$$\Delta t_2' = \Delta t_2 + t(\delta l, \phi_2)$$

Finally, for the differences of differences, we find...

$$\Delta t_1' - \Delta t_2' = \Delta t_2 - \Delta t_1 + (t(\delta l, \phi_2) - t(\delta l, \phi_1))$$
$$\Delta t_1' - \Delta t_2' = \frac{l\beta^2}{c}(\cos 2\phi_1 - \cos 2\phi_2) + \frac{\delta l \beta^2}{c}(\sin {\phi_1}^2 - \sin {\phi_2}^2)$$
$$\Delta t_1' - \Delta t_2' = \frac{l\beta^2}{c}(\cos 2\phi_1 - \cos 2\phi_2) + \frac{\delta l \beta^2}{2c}(\cos 2\phi_2 - \cos 2\phi_1)$$
$$\Delta t_1' - \Delta t_2' = \frac{(2l + \delta l)\beta^2}{2c}(\cos 2\phi_1 - \cos 2\phi_2)$$

Thus, the change in the number of fringes is given by

$$\boxed{\delta = \frac{(l + \frac{\delta l}{2})}{\lambda}\beta^2(\cos 2\phi_1 - \cos 2\phi_2)}$$

As long as ϕ_2 ϕ_1 are different, then the interferometer would be sensitive to the velocity of order β^2, unless, of course, the speed of light is independent of the speed of the source.

(c) In the Michelson-Morley experiment $\delta l = 0$, $\phi_1 = 0$, and $\phi_2 = \pi/2$, so we find...

$$\boxed{\delta = \frac{2l}{\lambda}\beta^2}$$

(d) You just showed that slight changes in δl and ϕ are proportional to β. As such, if β is zero, the change in the number of fringes also remains zero. So, what is important here? What's important is that the apparatus does not change as one goes from Δt_1 to Δt_2. For example, the Kennedy-Thorndike experiment had the experiment remain still while the Earth moved around (in orbit around the sun or around it's own axis). Aside from temperature and vibrational modulations, the experiment would be fairly immune to changes in the path length, providing a robust measure of the ether wind, if it existed.

Problem 4: Getting All Your Clocks in a Row... [20 pts]

Suppose there are 5 clocks sitting on a train platform, all syncronized with each and each clock is 1 meter apart from the next. A passing train, moving at velocity $v = c/3$, has the identical setup. Two observers, one on the train (Zak) and one on the platform (Jill), have the confusing assignment of recording the time registered on each of these clocks (a UROP project that went horribly wrong...). Both Zak and Jill are located next to the center clock. They pass each other at $t = t' = 0$.

(a) Suppose Jill records the status of the clocks on the train when all her clocks read t=0. What does she observe? Draw what she sees.

(b) Suppose Zak records the status of the clocks on the platform when all his clocks register t'=0. What does he observe? Draw what he sees.

(c) How do you reconcile their observations?

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(a) We need to keep in mind that clocks that may be synchronized in one frame *are not necessarily so in another.* Einstein emphasized this concept in his 1905 paper. I run into issues once I try to match my clock to a different clock at a different position with a relative velocity to my frame. In trying to keep the speed of light constant for all observers (Einstein's second postulate), I find that synchronization in one frame is no longer guaranteed in any other frame. Luckily, I can use the Lorentz transformations to tell me how to interpret what I observe.

The Lorentz transformation allows one to correctly interpret events from one frame to the other. This is given by the formula in class...

$$t' = \gamma(t - vx/c^2)$$
$$x' = \gamma(x - vt)$$

and the inverse transformation...

$$t = \gamma(t' + vx'/c^2)$$
$$x = \gamma(x' + vt')$$

In our case, we are told at $t = t' = 0$ the two center clocks coincide ($x' = x = 0$). What about the other clocks? Let us call Zak's frame (the train) S' and Jill's frame (on the platform) S. Now, Zak looks at his clocks and claims that they are synchronized in his frame. How does he synchronize them? He could send light pulses to the left and right clocks and adjust for the time it takes for the light to propagate there (and back). Since all of Zak's clocks are in the same frame, he finds that the light pulses arrive to the left and right clocks at the same time. Ok, everything is fine so far.

Jill watches the same scene from the platform, with Zak's train zipping along from left to right. She sees Zak send out his pulses, but the ones going to the left (remember, those pulses always travel at speed c, no matter the speed of the source) arrive sooner than those going to the right! What was synchronized in one frame is no longer so in the other. The Lorentz transformations allow us to provide a quantitative description of the said effect in a way that is consistent with postulates 1 and 2.

In the first part of the problem, we have the clocks synchronized in the S' frame. That is, $t'_{-2} = t'_{-1} = t'_0 = t'_1 = t'_2 = 0$. To convert to the S frame (what Jill sees on her platform), we use the Lorentz transformation...

$$\gamma = \frac{1}{\sqrt{1-(1/3)^2}} = \frac{3}{2\sqrt{2}}$$
$$t_i = \gamma(t'_i + vx'_i/c^2)$$
$$t_i = \frac{1}{2\sqrt{2}}\frac{x'_i}{c}$$

The results are shown in Table **??**.

Clock Number	-2	-1	0	1	2
Time t' (ns)	0	0	0	0	0
Position x' (m)	-2	-1	0	1	2
Time t (ns)	-2.36	-1.18	0	1.18	2.36

(b) In this case, we have the system exactly reversed. Zak now is at rest with his clocks (which he has syncronized in his frame) and he makes the observation of what the clocks on the platform read. Everything is identical to the previous problem, except Jill and her clocks are moving in the opposite direction (right to left). Hence, as expected, the reverse transform inherits a minus sign...

$$t'_i = \gamma(t_i - vx_i/c^2)$$
$$t'_i = -\frac{1}{2\sqrt{2}}\frac{x_i}{c}$$

Clock Number	-2	-1	0	1	2
Time t (ns)	0	0	0	0	0
Position x (m)	-2	-1	0	1	2
Time t (ns)	2.36	1.18	0	-1.18	-2.36

Notice the clocks now point "outward" rather than "inward".

(c) The main point is that clocks in sync in one frame are not in-sync in another frame. There is no "right frame" in this picture, they are both right, for they are describing different situations in each case.

Problem 5: Review: Space-Time Diagrams [20 pts]

Although events (occurrences in space-time) do not care what coordinate system one uses, in practical terms we always employ some coordinate system to describe the location of an event with respect to some other.

Let us start by looking at a more familiar way to represent coordinates, the x-y plane. Consider a point in this plane at some distance from the origin (say, p=(x_0, y_0)). Now, imagine that I decide to rotate my coordinate system by some angle, ϕ. My new coordinate system would be described as follows...

$$x' = x\cos\phi + y\sin\phi$$
$$y' = -x\sin\phi + y\cos\phi$$

...or, in matrix notation...

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Begin by drawing the x and y axis for both the non-rotated and rotated frames.

(b) Draw the point p from above and project where the point lands on each of the axes.

(c) Show that the *distance* from the origin remains unchanged in the x'-y' coordinate system.

(d) Show that $R^T R$ is unitary, where R is the above rotation matrix.

Lorentz transformations can also be considered a type of rotation, although here the rotation is between the spatial and time coordinates. Consider then a Lorentz boost along the x-axis (for simplicity, we consider just the x-t axis).

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

where $\tanh\phi = v/c$.

(e) Draw the x and ct axis for both the boosted and non-boosted frames.

(f) Now consider a point $q = (ct_0, x_0)$. Draw this point and project where the point lands on each of the axes.

(g) What strikes you as different about boost versus rotation?

(h) Let us now consider the "distance" of q from the origin. Show that if the distance in the (t, x) plane were to be defined in the same way as that in the (x, y) plane (i.e. $d^2 = (ct^2) + x^2$), then it loses invariance between the unboosted and boosted coordinate systems. How would you modify the definition of this "distance" to keep it invariant under boosts?
[Hint: Use the identity $\sinh^2\phi - \cosh^2\phi = -1$.]

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(a) To draw the proper axes under this rotation, let us recall how to draw axes in the first place. For the x-y plane, we draw all points for which x=0 is true for the y-axis, and we draw the set of point where y=0 for the x-axis. We repeat the same procedure for the x'-y' plane.

$$\begin{aligned} \text{For the y} - \text{axis...} \\ x' = 0 = x\cos\phi + y\sin\phi \\ \to y = -\cot\phi\ x = \tan(\phi - \pi/2) \\ \text{For the x} - \text{axis...} \\ y' = 0 = -x\sin\phi + y\cos\phi \\ \to y = \tan\phi\ x \end{aligned}$$

Note that the x and y axis are still 90 degrees offset from one another, while the axes themselves have tilted by a small angle ϕ.

(b) Though the point doesn't move, the projection along the axes obviously does.

(c) The distance to the origin (the rotation does not change the location of the origin, hence the origin is shared in common with both frames) is given by...

$$\begin{aligned} (r')^2 = (x')^2 + (y')^2 \\ (r')^2 = (x\cos\phi + y\sin\phi)^2 + (-x\sin\phi + y\cos\phi)^2 \\ (r')^2 = x^2\cos\phi^2 + y^2\sin\phi^2 + 2xy\cos\phi\sin\phi + x^2\sin\phi^2 + y^2\sin\phi^2 - 2xy\cos\phi\sin\phi \\ (r')^2 = x^2(\cos\phi^2 + \sin\phi^2) + y^2(\cos\phi^2 + \sin\phi^2) \\ (r')^2 = x^2 + y^2 \\ (r')^2 = r^2 \end{aligned}$$

So rotation has no effect on the distance of the point to the origin (it also preserves the length of the object as well).

(d) Let us write out explicitly R and R^T...

$$R = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$
$$R^T = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$
$$R^T R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

$$R^T R = \begin{pmatrix} \cos\phi^2 + \sin\phi^2 & \cos\phi\sin\phi - \cos\phi\sin\phi \\ \cos\phi\sin\phi - \cos\phi\sin\phi & \cos\phi^2 + \sin\phi^2 \end{pmatrix}$$
$$R^T R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(e) We use the same procedure as in part (a)...

$$\text{For the x} - \text{axis...}$$
$$ct' = 0 = ct\cosh\phi - x\sinh\phi$$
$$\rightarrow ct = \tanh\phi \; x = \beta \; x$$
$$\text{For the ct} - \text{axis...}$$
$$x' = 0 = -ct\sinh\phi + x\cosh\phi$$
$$\rightarrow ct = \coth\phi \; x = \frac{1}{\beta} \; x$$

(f) The important thing to keep in mind here is that the line connecting point q and its projection on one axis has to be parallel to the other axis. The rest is intuitive.

(g) What is perhaps striking about Lorentz boosts is that time and space (x) are no longer orthogonal to one another. In fact, as β approaches 1 (speed of light), the x-axis and the time axis converge.

(h) We want to calculate the distance from the origin to q in both coordinates. Using the Lorentz transformation given by the problem, we have:

$$ct' = \cosh\phi \, ct - \sinh\phi \, x$$
$$x' = -\sinh\phi \, ct + \cosh\phi \, x.$$

If we use the definition of distance in the (x, y) plane, we'd have:

$$\begin{aligned}
& (ct')^2 + x'^2 \\
= \; & (\cosh\phi\, ct - \sinh\phi\, x)^2 + (-\sinh\phi\, ct + \cosh\phi\, x)^2 \\
= \; & \cosh^2\phi\, c^2t^2 - 2\sinh\phi\cosh\phi\, ctx + \sinh^2\phi\, x^2 \\
& + \cosh^2\phi\, x^2 - 2\sinh\phi\cosh\phi\, ctx + \sinh^2\phi\, c^2t^2 \\
= \; & (\cosh^2\phi + \sinh^2\phi)(c^2t^2 + x^2) - 4\sinh\phi\cosh\phi\, ctx.
\end{aligned}$$

This is clearly not invariant.

So instead, we define the distance as $d^2 = (c^2t^2) - x^2$, and now:

$$\begin{aligned}
& (ct')^2 - x'^2 \\
= \; & (\cosh\phi\, ct - \sinh\phi\, x)^2 - (-\sinh\phi\, ct + \cosh\phi\, x)^2 \\
= \; & \cosh^2\phi\, c^2t^2 - 2\sinh\phi\cosh\phi\, ctx + \sinh^2\phi\, x^2 \\
& - \cosh^2\phi\, x^2 + 2\sinh\phi\cosh\phi\, ctx - \sinh^2\phi\, c^2t^2 \\
= \; & (\cosh^2\phi - \sinh^2\phi)(c^2t^2 - x^2) \\
= \; & c^2t^2 - x^2
\end{aligned}$$

becomes invariant again.

Note that it could also work if we would have defined our distance as $d^2 = -(c^2t^2) + x^2$.

The general mathematical object that relates coordinates and distances of a set of points is called the metric tensor. The set of these points would then form a metric space (you can check the precise definition for mathematical rigor). For special relativity, we would always use the Minkowski metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$ to model the spacetime as a Minkowski space (You might remember from our lecture on the history of relativity that Hermann Minkowski was one of Einstein's instructors at ETH.)