(12) Damped Oscillators

We've been trying to ignore it, but in the real world there is friction. Friction means that mechanical energy is converted to thermal energy, and we no longer have a 'conservative' system. But we can try.

Imagine some fraction of kinetic energy is couple to thermal energy per unit time β.

$$L'\left(q,\dot{q}\right) = \left(T + T_{\mu}\right) - U = L + \int \beta T dt \tag{1}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d}{dt}\left(\int \frac{\partial}{\partial \dot{q}}\left(\epsilon T\right) dt\right) \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial}{\partial \dot{q}}\left(\epsilon T\right) = \frac{\partial L}{\partial q} \end{aligned}$$

Generalizing ϵT to any velocity dependent function,

Let

$$D = \frac{1}{2}\sum_{j,k} b_{jk}\dot{q}_j\dot{q}_k \quad \text{'dissipative function'} \tag{2}$$

or, for one degree of freedom, write

$$D = \frac{1}{2}b\dot{q}^2 \tag{3}$$

write

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} - \frac{\partial D}{\partial \dot{q}} \tag{4}$$

such that

$$\dot{p} = \left(\frac{\partial L}{\partial q} - b\dot{q}\right) \tag{5}$$

where $\frac{\partial L}{\partial q}$ is the conservative force, and $b\dot{q}$ is the dissipative force.

Let's put this to work on our harmonic oscillator to make a more realistic damped oscillator.

for

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{6}$$

and

$$D = \frac{1}{2}b\dot{x}^2 \tag{7}$$

$$m\ddot{x} = -kx - b\dot{x}, \qquad \omega_0^2 = \frac{k}{m} \tag{8}$$

or

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0, \qquad 2\lambda = \frac{b}{m} \tag{9}$$

This differential equation is best solved with complex exponentials, but the solution can be written in real form as

"under damped"

$$x\left(t\right) = ae^{-\lambda t}\cos\left(\omega t + \varphi\right) \text{ for } \lambda < \omega_0 \tag{10}$$

with

$$\omega = \sqrt{\omega_0^2 - \lambda^2} \tag{11}$$

"over damped"

$$x\left(t\right) = e^{-\lambda t}\left(a_1 e^{\beta t} + a_2 e^{-\beta t}\right) \text{ for } \lambda < \omega_0 \tag{12}$$

with

$$\beta = \sqrt{\lambda^2 - \omega_0^2}, \quad \text{note } \beta < \lambda \rightarrow \text{ decay} \tag{13}$$

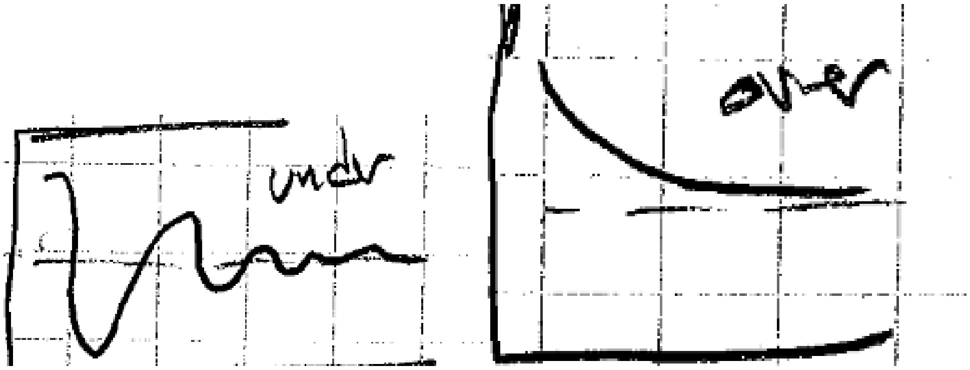


Figure 1: Plots of displacement versus time for an underdamped and overdamped harmonic oscillator, respectively.

“critically damped”

$$x\left(t\right) = e^{-\lambda t}\left(a_1 + a_2 t\right) \quad \text{for} \quad \lambda = \omega_0 \tag{14}$$

Damped systems lose energy with time until they come to rest. The rate of energy loss is given by the dissipation function.

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt} \quad \dot{q}\frac{\partial L}{\partial \dot{q}} - L \\
&= \dot{q}\frac{d}{dt} \quad \frac{\partial L}{\partial \dot{q}} \quad + \frac{\partial L}{\partial \dot{q}}\ddot{q} - \quad \frac{\partial L}{\partial t} + \dot{q}\frac{\partial L}{\partial q} + \ddot{q}\frac{\partial L}{\partial \dot{q}} \\
&= \dot{q} \quad \frac{d}{dt} \quad \frac{\partial L}{\partial \dot{q}} \quad - \frac{\partial L}{\partial q} \qquad \text{normally zero, but...} \\
\frac{dE}{dt} &= -\dot{q}\frac{\partial D}{\partial \dot{q}} = -2D
\end{aligned}$$

Note that the last line is just the rate of work done by friction as force × velocity.

To complete the picture, we should add a driving force to our damped oscillator. Returning to the equation of motion...

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = \frac{F(t)}{m} = \frac{f}{m}\cos(\omega t) \tag{15}$$

$$\rightarrow x(t) = a_1 e^{-\lambda t}\cos(\omega_0 t + \varphi) + a_2\cos(\omega t + \theta) \tag{16}$$

$$\begin{aligned} a_2 &= \frac{f}{m\ \sqrt{\left(\omega^2 - \omega_0^2\right)^2 + 4\lambda^2\omega^2}} \\ \tan\theta &= \frac{2\lambda\omega}{\omega^2 - \omega_0^2} \end{aligned}$$

where a_1 and φ come from the initial conditions

Again, the driven solution has 2 parts, one that depends on the initial conditions and another which is the response to the drive. With damping, we see that the first of these decays with time, such that the motion at $t \gg \frac{1}{\lambda}$ is essentially only the drive response.

$$x(t) \simeq a_2\cos(\omega t + \theta) \quad \text{for } t \gg \frac{1}{\lambda} \tag{17}$$

$$\rightarrow \frac{x(t)}{F(t)} = \frac{a_2}{f} \tag{18}$$

Let's try a slightly different pendulum system this time for our demo.

$$T = \frac{1}{2}m\ \ \dot{x}_m^2 + \dot{y}_m^2\ \ ,\ U = mgy_m,\ \ D = \frac{1}{2}b\dot{\varphi}^2 \tag{19}$$

$$x_m = l\sin\varphi\ \ ,\ \ y_m = y_d - l\cos\varphi \tag{20}$$

$$\dot{x}_m = l\cos\varphi\dot{\varphi}\ \ ,\ \ \dot{y}_m = \dot{y}_d + l\sin\varphi\dot{\varphi} \tag{21}$$

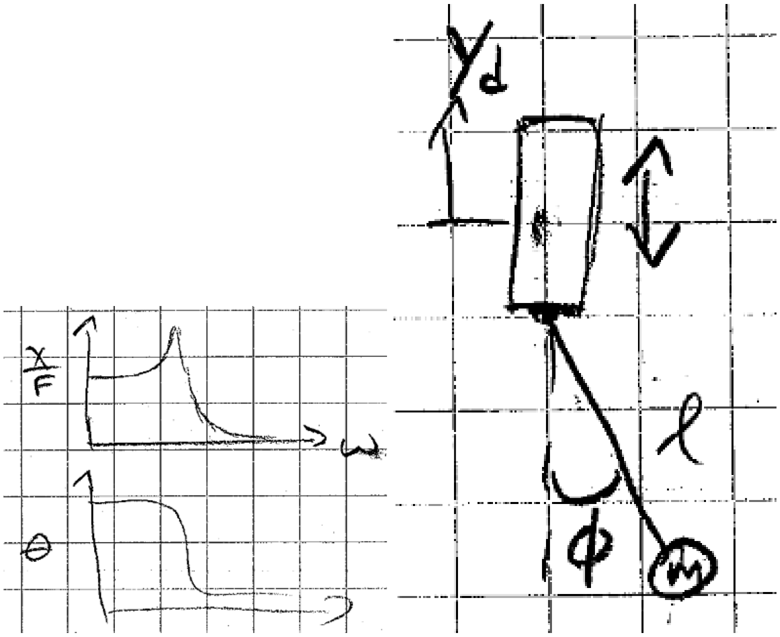


Figure 2: On the left are the TF plots, and on the right our new pendulum system.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = ml\left(l\ddot{\varphi} + \cos\varphi\dot{\varphi}\dot{y}_d + \sin\varphi\ddot{y}_d\right) \tag{22}$$

$$\frac{\partial L}{\partial \varphi} - \frac{\partial D}{\partial \dot{\varphi}} = ml\left(\cos\varphi\dot{\varphi}\dot{y}_d - g\sin\varphi\right) - b\dot{\varphi} \tag{23}$$

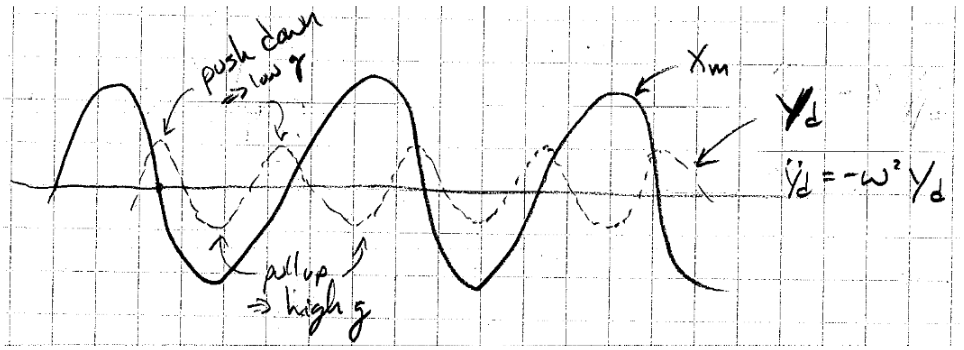
Which gives us the equation of motion.

A vertically driven pendulum is a bit of a strange things; it doesn't seem to work as a driver!

$$l\ddot{\varphi} + \frac{b}{lm}\dot{\varphi} + (g + \ddot{y}_d)\sin\varphi = 0 \tag{24}$$

where the damping term is $\frac{b}{lm}\dot{\varphi}$.

Instead, the drive appears to modify gravity. This makes sense, due to the equivalence principle. Interestingly, this lets us explore parametric resonance...



Notice how the pendulum becomes excited with a drive at twice the resonance frequency.

We won't cover parametric resonance further, but LL27 does. Methods for understanding non-linear/anharmonic behavior are also covered in LL 28-29, but I found the math unenelightening, so I won't try to reproduce it here.

We also see strange behavior for a high frequency drive. Damping is not important for this, so let's operate with $b = 0$. We can understand this by noticing that the pendulum's motion consists of a high frequency part (at the drive frequency) and a low frequency part (swinging around).

$$\varphi(t) \qquad \varphi_1(t) + \varphi_2(t) \tag{25}$$

where φ_1 corresponds to slow oscillations, and φ_2 to fast.

$$l\ddot{\varphi} + (g + \ddot{y}_d)\sin\varphi = 0 \tag{26}$$

$$l(\ddot{\varphi}_1 + \ddot{\varphi}_2) + (g + \ddot{y}_d)\sin(\varphi_1 + \varphi_2) = 0 \tag{27}$$

assume $\varphi_1 \sim$ const, and $\varphi_2 \ll 1$.

$$l\ddot{\varphi}_2 + (g + \ddot{y}_d)(\sin\varphi_1 + \cos\varphi_1\varphi_2) = 0 \tag{28}$$

for $y_d = a_d\cos(\omega t)$,

$$\ddot{y}_d = -a_d\omega^2\cos\omega t = -\omega^2 y_d \tag{29}$$

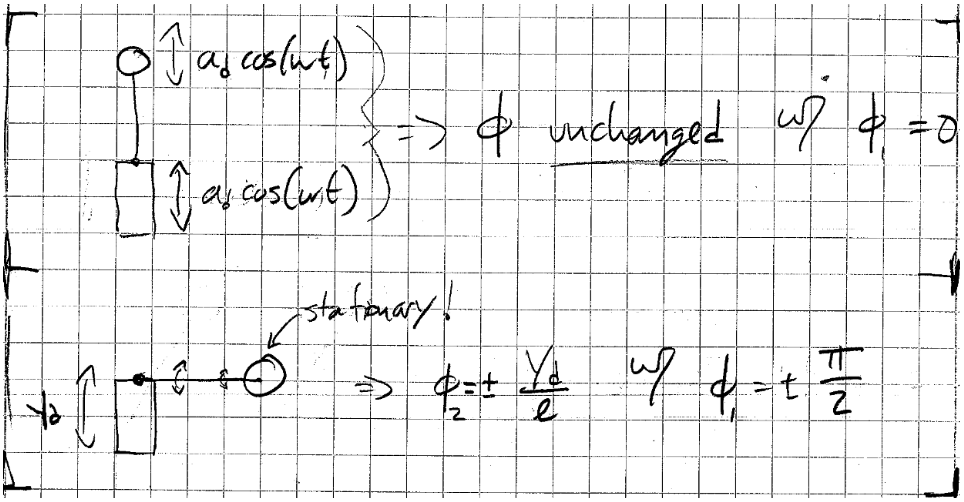


Figure 3: High frequency drive pendulum.

fast oscillation terms, to first order are

$$l\ddot{\varphi}_2 + g\cos\varphi_1\varphi_2 = a_d\omega^2\sin\varphi_1\cos\omega t \tag{30}$$

driven response:

$$\begin{aligned} \rightarrow \varphi_2 \qquad & \frac{a_d\omega^2\sin\varphi_1}{l\left(\omega_0^2\cos\varphi_1 - \omega^2\right)}\cos\omega t \\ & \sin\varphi_1\frac{y_d}{l} \quad \text{for } \omega \gg \omega_0 \end{aligned}$$

Graphically, this result is shown in Fig 3
Returning to our equations of motion, but keeping $\varphi_2 \ll 1$

$$l\ddot{\varphi}_1 - \omega^2\sin\varphi_1 y_d + \quad g - \omega^2 y_d \quad \left(\sin\varphi_1 + \cos\varphi_1\sin\varphi_1\frac{y_d}{l}\right) = 0 \tag{31}$$

$$l\ddot{\varphi}_1 + g\sin\varphi_1 = 2\omega^2\sin\varphi_1 y_d + \frac{\omega^2}{2l}\sin(2\varphi_1)y_d^2 \tag{32}$$

We are looking for the slow behavior, so let's average over the fast drive period, shown in Fig. 4.

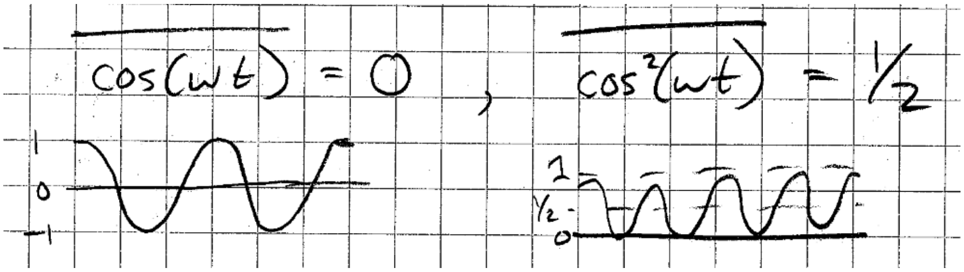


Figure 4: Averaging over the fast drive period.

$$\to \ddot{\varphi}_1 + \frac{g}{l}\sin\varphi_1 = \frac{1}{2} \quad \frac{a_d\omega}{l} \quad ^{2} \sin(2\varphi_1) \tag{33}$$

If we are close to φ_1 $\quad \pi$ (pointing up)

for $\varphi_1 = \pi + \varepsilon$ with $\varepsilon \ll 1 \to \sin\varphi_1 \quad -\varepsilon$

$$\to \ddot{\varepsilon} + \quad \frac{a_d\omega}{l} \quad ^{2} - \frac{g}{l} \quad \varepsilon = 0 \tag{34}$$

$$\to \quad \text{oscillator with } \omega_0^2 = \quad \frac{a_d\omega}{l} \quad ^{2} - \frac{g}{l} \tag{35}$$

stable if

$$a_d^2\omega^2 > gl \tag{36}$$

So, as we have seen, the Kapitza pendulum is stable around $\varphi \sim \pi$ (i.e. inverted) given a sufficiently fast drive.

Generally, when treating motion in a rapidly oscillating field, we can define an effective potential

$$U_{eff} = U + \frac{1}{2}m\dot{q}_f^2 \tag{37}$$

where $q_f(t)$ is the fast part of $q(t) = q_{slow} + q_{fast}$

for us, this would be

$$
\begin{aligned}
U_{eff} &= mgl\cos\varphi + \frac{1}{2}m\left(l\dot{\varphi}_2\right)^2 \\
&= mgl\cos\varphi + \frac{1}{2}m\left(\sin\varphi\dot{y}_d\right)^2 \\
&= mgl\cos\varphi + \frac{1}{2}m\sin\varphi^2 \quad \frac{1}{2}a_d\omega \quad {}^{2}
\end{aligned}
$$

8.223 Classical Mechanics II
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