

Massachusetts Institute of Technology  
Physics 8.224. Black Holes and Astrophysics  
Spring 2003

MIDTERM EXAMINATION

This exam is CLOSED BOOK; no printed materials are allowed. You may consult ONE 8.5 by 11 inch sheet of paper of handwritten notes using both sides of the paper. Put your name on this sheet of notes and turn it in with the completed examination.

Attached is a photocopy of the Selected Formulas that face the inside back cover of the text *Exploring BHs*. You may request any physical constants needed to complete the examination, such as the speed of light or the radius of Earth. You may request that the Kerr metric be written on the board.

You may NOT use ANY electronic device during the examination except for a non-computer watch.

DO ALL WORK IN THIS EXAMINATION BOOKLET.

*Suggestion:* Write all preliminary notes on the back of the facing page. This may allow us to award partial credit.

Part I. 9 points	
Part II. 34 points (correct minus incorrect)	
Part III. 20 points	
Part IV. 21 points	
Part V. 16 points + 8 extra	
TOTAL 100 points + 8 extra	

**PART I. QUESTION FROM THORNE (9 points)**

Using one to three sentences for each, answer the following questions

- A. Why does our sun not collapse into a black hole?
- B. Why does a white dwarf not collapse into a black hole?
- C. Why does a neutron star not collapse into a black hole?

**PART II. MULTIPLE-SELECT QUESTIONS (34 POINTS)**

Put T (true) or F (false) to the left of every statement A, B, C, . . .

More than one statement in a given set may be correct.

The score will be number of correct answers minus the number of incorrect answers. Blank items will be marked incorrect. Therefore guess when you are unsure.

**1. A rocket ship is at rest near a spinning black hole but outside the static limit.**

- A. This ship has zero angular momentum with respect to the center of the black hole.
- B. This rocket ship must fire its rocket to remain stationary.
- C. The rocket blast must have a tangential component in order for the rocket ship to remain stationary.
- D. The view of the fixed stars seen by eye from the rocket ship does not change with time.
- E. The view of the fixed stars is the same as the view of the same stars seen by an observer much farther out along the same fixed radius.
- F. A distant observer launches a tennis ball radially inward. By chance, this tennis ball passes close to the stationary rocket ship. The rocket ship observer sees the tennis ball as falling in the radially inward direction.

**2. The following statements describe the motion of a light flash near a NON-spinning black hole, but outside the horizon (unless otherwise noted).**

- A. The bookkeeper speed of light is not measured directly by any local observer close to the black hole.
- B. The shell observer measures the speed of light to be unity (in meters per meter).
- C. The deflection of a light flash depends upon its energy. For given initial location and direction, a gamma ray flash will be deflected differently than a flash of visible light.
- D. Question: Why does a flash of light that passes our sun take longer to traverse its path to a distant observer than the same flash in the absence of the sun?  
Answer: Because the light path is deflected and therefore has a sufficiently longer path to account for the longer transit time.
- E. Light can follow a stable circular orbit around a non-spinning black hole.
- F. An observer falling toward the center *inside* the horizon cannot see the crunch point ahead of her.

**3. The following statements concern the Global Positioning System (GPS).**

- A. The gravitational deflection of the trajectories of radio waves between orbiting satellite and earth-surface receiver has a significant effect on the operation of the GPS.
- B. The orbits of the satellites can be calculated sufficiently accurately using Newtonian mechanics for the purpose of operating the GPS.
- C. The receiver on earth's surface requires signals from TWO GPS satellites in order to locate its position.
- D. The satellite clock rates are significantly affected by both the relative speed of satellite and receiver and the difference in radii between satellite and receiver.
- E. It would be possible in principle to design and install a GPS with the satellites parked in "stationary" orbits around the equator, each satellite permanently above a fixed point on earth's surface. (But such a system might not be useful at locations near the poles.)

**4. The following statements concern various kinds of coordinates for locating events near a NON-spinning black hole. All clocks etc are outside the horizon.**

- A. All shell clocks on a given spherical shell can be synchronized with one another and will remain synchronized.
- B. A clock very far from the black hole can display bookkeeper time directly.
- C. In general, whatever the metric, bookkeeper coordinates of an event do not necessarily refer by themselves to physical measurements with rods and clocks.
- D. A shell observer and an observer passing the shell in free-fall can compare their local coordinate values of nearby events using the Lorentz transformation of special relativity.
- E. A shell clock "runs slow" compared with a far-away clock. QUESTION: Is this due to red shift of the connecting pulses or actual slowness of the shell clock? ANSWER: There is no physical way to tell the difference!

**5. According to the Friedmann-Robertson-Walker cosmological model:**

- A. It is possible for the Universe to be closed and expand forever.
- B. It is possible for the Universe to be open and expand forever.
- C. It is possible for the Universe to be flat and expand forever.
- D. It is possible for the Universe to be closed and recontract.
- E. It is possible for the Universe to be open and recontract.
- F. It is possible for the Universe to be flat and recontract.
- G. It is possible for the Universe to enter a static state, but that state is unstable.

**6. The following statements concern the supermassive black hole at the center of our galaxy.**

- A. We can observe motions of stars near the black hole using frequencies of light in the visible region of the electromagnetic spectrum.
- B. X-rays can be focused using glancing angles off a tapered cylindrical reflecting surface.
- C. The fact that the intensity of X-radiation from the center of the galaxy often changes by factors of 3 or 4 in ten minutes implies that the emitting source is not much larger than 10 light-minutes in spatial dimension.
- D. An apparent X-ray jet from the black hole and X-ray lobes on either side of the black hole may tell us about past ejections or outflows from the supermassive black hole.
- E. The approximate mass of the black hole *cannot* be inferred from trajectories of nearby stars.

**HAVE YOU MARKED AS "TRUE" ALL THE CORRECT STATEMENTS IN EACH SET? (IN EACH SET THERE MAY BE ONE, MORE THAN ONE, OR POSSIBLY NONE CORRECT)**

### PART III. The Robertson-Walker metric for the Universe (20 points)

Here is the spacelike form of the Robertson-Walker metric for the Universe.

$$d\sigma^2 = dt^2 + R^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right]$$

where  $R(t)$  is the scale factor and  $K$  is the curvature factor. Consider the resulting metric at the present time  $t_0$ , so that the scale factor  $R(t) = R(t_0) = 1$ . Choose an arbitrary point as the center ( $r = 0$ ) of your coordinate system. The following questions concern two events near to one another in space that occur simultaneously ( $dt = 0$ ) in the frame described by the metric.

- A. (4 points) Show that very close to the origin (close to  $r = 0$ ), the proper distance between these two events is described by Euclidean geometry, no matter what the value of  $K$ , the curvature factor.
- B. (4 points) For  $K > 0$  (positive curvature) and  $r$  remote from the origin, is the proper distance between these two events greater than, equal to, or less than the distance reckoned using Euclidean geometry? Demonstrate your answer using the metric.
- C. (4 points) For  $K < 0$  (negative curvature) and  $r$  remote from the origin, is the proper distance between these two events greater than, equal to, or less than the distance reckoned using Euclidean geometry? Demonstrate your answer using the metric.
- D. (8 points) For the conditions of part B ( $K > 0$  and large  $r$ ) what is the largest possible value of  $r$ ? This may be either a coordinate singularity or a horizon. To find out which, change variables by writing

$$r = (\sin u) / \sqrt{K}$$

Rewrite the metric in terms of  $dt$  and  $du$  and  $d\Omega$ . Is the maximal  $r$  an event horizon or simply a coordinate singularity? Will an observer at this value of  $r$  be UNABLE to measure the separation of nearby simultaneous events in all directions equally?

**PART IV. Launched Stone (21 points)**

A stone of mass  $m$  has total energy  $(5/3)m$  as measured by a shell observer at radius  $r = 6M$  in a Schwarzschild spacetime, where  $M$  is the mass of the black hole. The stone is moving initially at a 30-degree angle to the radially *outward* direction as measured by the shell observer. You are given

$$\sin 30^\circ = 1/2 \text{ and } \cos 30^\circ = \sqrt{3}/2$$

(Leave any square roots in answers in square root form.)

You may also need the equation relating the shell energy  $E_{\text{shell}}$  and energy-at-infinity  $E$  for *anything* moving near a nonspinning black hole. (Equation [27], page 3-17 of EBH)

$$E = \left[ \frac{1}{\sqrt{1 - \frac{2M}{r}}} \right] E_{\text{shell}}$$

- A. Find the bookkeeper expressions for  $dr/dt$  and  $d\phi/dt$  for the stone in terms of  $M$ , the mass of the black hole.
- B. Find the values of the energy-at-infinity  $E$  and angular momentum  $L$  of the stone in terms of  $m$  and  $M$ .
- C. GIVEN that the launch radius is greater than the radius of the peak of the effective potential, will the stone fall into the black hole or not? Explain your answer.

**PART V. A Static, Spherically Symmetric Spacetime (16 points+ 8 extra credit)**

A certain spherically symmetric, static spacetime has the metric

$$ds^2 = (1 - Kr^2) dt^2 - (1 - Kr^2)^{-1} dr^2 - r^2 d\Omega^2$$

where  $K$  is a positive constant. The Euler-Lagrange equations are:

$$\frac{d}{d\lambda} \left( \frac{\partial f}{\partial \dot{x}^\mu} \right) - \frac{\partial f}{\partial x^\mu} = 0$$

- A.** Using the calculus of variations and the Euler-Lagrange equations, obtain expressions for the conserved energy per unit mass  $E/m$  and angular momentum per unit mass  $L/m$ .
- B. (extra credit 5 points).** Use the expressions for  $E/m$  and  $L/m$  to eliminate  $dt$  and  $d\Omega$  in the metric and thereby obtain an expression for  $V_{\text{eff}}^2(r)$  in the radial equation

$$\left( \frac{dr}{d\lambda} \right)^2 = \frac{E^2}{m^2} - \frac{V_{\text{eff}}(r)}{m^2}$$

- C. (extra credit 3 points).** Qualitatively describe the motion of a particle released from rest at radius

$$r = 1/(2\sqrt{K})$$