Lecture 18 - Topics

- Open Strings

Still for open string:

Heisenberg operators: $X^I(\tau,\sigma)$, x_0^-, $\mathcal{P}^{\tau I}(\sigma)$, p^+

$$[X^I(\sigma), \mathcal{P}^{\tau J}(\tau,\sigma')] = i\eta^{IJ}\delta(\sigma-\sigma')$$
$$[x_0^-, p^+] = -i$$

$$\frac{\partial}{\partial\tau} = 2\alpha' p^+ + \frac{\partial}{\partial X^+} \Leftrightarrow \underbrace{2\alpha' p^+ p^-}_{\text{Hamiltonian, } H} = H$$

$$p^- = \int d\sigma(\mathcal{P}^{-\tau} = \frac{1}{2\pi\alpha'}\frac{\partial X^-}{\partial\tau})$$

$H = 2\alpha' p^+ p^- = L_0^+$ from analysis of classical string

Are we *sure* $H = 2\alpha' p^+ p^-$? After all, p^- is the product of lots of operators, which can be ill-defined. Must be careful in our quantum case.

$$\ddot{X}^I - X^{I''} = 0$$

$$X^I(\tau,\sigma) = x_0^I + \sqrt{2\alpha'}\alpha_0^I\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I\cos(n\sigma)e^{-in\tau}$$

$$\mathcal{P}^{\tau J} = \frac{1}{2\pi\alpha'}\frac{\partial x^J}{\partial\tau}$$

$$(\dot{X}^I + X^{I\prime})(\tau,\sigma) = \sqrt{2\alpha'}\sum_{n\in Z}\alpha_n^I e^{(-in(\tau+\sigma))} \qquad \sigma\in[0,\pi] \tag{1}$$

$$(\dot{X}^I - X^{I\prime})(\tau,\sigma) = \sqrt{2\alpha'}\sum_{n\in Z}\alpha_n^I e^{(-in(\tau-\sigma))} \qquad \sigma \in [0,\pi] \tag{2}$$

This is an important computation. Later, we will do this for closed strings too, and we'll see very similar (though not same).

Best way to select Fourier modes is in $[0, 2\pi]$ but $\sigma \in [0,\pi]$. $\sigma \to -\sigma$

$$(\dot{X}^I - X^{I\prime})(\tau,-\sigma) = \sqrt{2\alpha'}\sum_{n\in Z}\alpha_n^I e^{(-in(\tau+\sigma))} \tag{2'}$$

This makes sense when $\sigma \in [-\pi, 0]$.

$$\begin{aligned} A^I(\tau,\sigma) &= \sqrt{2\alpha'}\sum_{n\in Z}\alpha_n^I e^{(-in(\tau+\sigma))} \qquad \sigma \in [-\pi,\pi] \\ &= \begin{cases} (\dot{X}^I + X^{I\prime})(\tau,\sigma) & \sigma\in[0,\pi] \\ (\dot{X}^I - X^{I\prime}) & \sigma \in [-\pi, 0] \end{cases} \end{aligned}$$

Now have σ defined over $[-\pi,\pi]$.

$$\begin{gathered} [X^I(\tau,\sigma), \dot{X}^I(\tau,\sigma')] = 2\pi\alpha' i\eta^{IJ}\delta(\sigma-\sigma') \\ [\dot{X}^I(\tau,\sigma), \dot{X}^J(\tau,\sigma')] = 0 \end{gathered}$$

$[X^{I\prime}(\tau,\sigma), X^{J\prime}(\tau,\sigma')] = 0$ X's commute at different σ's so can then differentiate.

$$\begin{aligned} [(\dot{X}^I \pm X^{I\prime})(\tau,\sigma), (\dot{X}^J \pm X^{J\prime})(\tau,\sigma)] &= [(\dot{X}^I \pm X^{I\prime})(\tau,\sigma), (\dot{X}^J \pm X^{J\prime})(\tau,\sigma)] \\ &= \pm 4\pi\alpha' i\eta^{IJ}\frac{d}{d\sigma}(\sigma-\sigma') \end{aligned}$$

$$[A^I(\tau,\sigma), A^J(\tau,\sigma')] = 2\alpha' \sum_{m',n'} e^{(-im'(\tau+\sigma))} e^{(-in'(\tau+\sigma'))} [\alpha^I_{m'}, \alpha^J_{n'}]$$
$$= \begin{cases} 4\pi\alpha' i\eta^{IJ} \frac{d}{d\sigma}\delta(\sigma-\sigma') & \sigma,\sigma' \in [0,\pi] \\ 4\pi\alpha' i\eta^{IJ} \frac{d}{d\sigma}\delta(\sigma-\sigma') = 0 & \sigma \in [0,\pi], \sigma' \in [-\pi, 0] \\ -4\pi\alpha' i\eta^{IJ} \frac{d}{d(-\sigma)}\delta(\sigma'-\sigma) = 4\pi\alpha' i\eta^{IJ} \frac{d}{d(\sigma)}\delta(\sigma-\sigma') & \sigma,\sigma' \in [-\pi, 0] \end{cases}$$

$$\boxed{\sum_{m',n'} e^{(-im'(\tau+\sigma))} e^{(-in'(\tau+\sigma'))} [\alpha^I_{m'}, \alpha^J_{n'}] = 2\pi i\eta^{IJ} \frac{d}{d\sigma}\delta(\sigma-\sigma') \qquad \sigma,\sigma' \in [-\pi,\pi]}$$

Apply the following integral operations:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\sigma e^{(im\sigma)} \cdot \frac{1}{2\pi}\int_{-\sigma}^{\sigma} d\sigma' e^{(in\sigma)}$$

Divide by $e^{(-i(m+n)\tau)}$ on both sides:

$$[\alpha^I_m, \alpha^I_n] = -n\eta^{IJ}\delta_{m+n,0} e^{(i(m+n)\tau)}$$
$$\boxed{[\alpha^I_m, \alpha^I_n] = m\delta_{m+n,0}\eta^{IJ}}$$

Commutation relation proved in book:

$$[x^I_0, p^J] = i\eta^{IJ}$$

Note:

$$\alpha^I_0 = \sqrt{2\alpha'} p^I$$

$$[\alpha^I_m, \alpha^I_n] = m\eta^{IJ}\delta_{m,n}$$
$$\alpha^\mu_n = a^\mu_n \sqrt{n} \qquad n > 0$$

$$\alpha^\mu_{-n} = a^{\mu+}_n \sqrt{n} = (\alpha^\mu_{+n})^+ \qquad n < 0$$

Opposite signs for m and n

$$[a^I_m, a^J_n] = 0$$
$$[a^{I+}_m, a^{J+}_n] = 0$$

$m > 0$,$n > 0$:

$$[a^I_m \sqrt{m}, a^J_n \sqrt{n}] = m\eta^{IJ}\delta_{m,n}$$
$$\boxed{[a^I_m, a^{J+}_m] = \eta^{IJ}\delta_{m,n}}$$

$$\sqrt{2\alpha'}\alpha^-_n = \frac{1}{p^+}L^\perp_n \overset{n=0}{\rightarrow} 2p^+p^- = \frac{1}{\alpha'}L^\perp_0$$

$$L^\perp_n = \frac{1}{2}\sum_{p\in Z}\alpha^I_{n-p}\alpha^I_p$$

Don't have to worry if $n \neq 0$. Might have to worry if $n = 0$.

But what we want is: $H = L^\perp_0 = 2\alpha' p^+p^-$. $L^\perp_0 = \frac{1}{2}\sum_{p\in Z}\alpha^I_{-p}\alpha^I_p$ but α's don't commute so don't know if this is right.

$$M^2 = -p^2 = 2p^+p^- - p^Ip^I = \frac{1}{\alpha'}L^\perp_0 - p^Ip^I$$

$$\begin{aligned} L^\perp_0 &= \frac{1}{2}\alpha^I_0\alpha^I_0 + \frac{1}{2}\sum_{p=1}^{\infty}(\alpha^I_{-p}\alpha^I_p + \alpha_p\alpha^I_{-p}) \\ &= \alpha' p^Ip^I + \sum_{p=1}^{\infty}\alpha_{-p}\alpha^I_p + \frac{1}{2}(D-2)\sum_{p=1} p \end{aligned}$$

Note $\alpha_{p>0}$ is destruction operation convention. $\alpha_{p<0}$ is creation operation convention.

$$\boxed{M^2 = \frac{1}{\alpha'}\left(\sum_{p=1}^{\infty} p a_p^{I+} a_p^I + \frac{1}{2}(D-2)\sum_{p=1}^{\infty} p\right)}$$

In classical theory, had

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} n a_n^{I+} a_n^I + \frac{1}{2}(D-2)\sum_{p=1}^{\infty} p\right)$$

Showed all states of string had mass ¿ 0. Couldn't get anything intersting without mass.

Would be great here if $\frac{1}{2}(D-2)\sum_{p=1}^{\infty} p = -1$. Then:

$$M^2 = \frac{1}{\alpha'}(\sum n a_n^{I+} a_n^I - 1)$$

Now want oscillation states without mass

$$\sum_{p=1}^{\infty} p = 1 + 2 + 3 + 4 + \ldots = -\frac{1}{12}$$

Crazy, huh? Not true in general, of course, but almost true in one sense. Since we want:

$$\frac{1}{2}(D-2)\sum_{p=1}^{\infty} p = -1$$

$$\frac{1}{2}(D-2)\left(-\frac{1}{12}\right) = -1 \Rightarrow D = 26(\text{dimension of string})$$

Now how is $\sum_{p=1}^{\infty} p = -\frac{1}{12}$?!

Recall Riemann Zeta Function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(s=-1) = -\frac{1}{12} = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n$$

$\zeta(s)$ well-defined and convergent for $s \geq 2$. Doesn't converge for $s = 1$(pole). ζ defined on *complex* plane.

The beauty of analytic functions: If you know it is defined in a very small finite regin, you know it everywhere by the Cauchy-Riemann.

$$\boxed{2p^+p^- = \frac{1}{\alpha'}(L_0^\perp + a) \qquad a = \text{constant}}$$

Define for once and for all:

$$\boxed{L_0^\perp = \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{p=1}^{\infty} \alpha_{-p}^I \alpha_p^I}$$

$$[M^{-I}(a, D), M^{-I}(a, D)] = 0$$

Set standards of *messy* computation. All books omit at least some details.

$$M^{-J} \approx \alpha_n^- \alpha_m^J \approx [L_n^+, L_m^+] = (m-n)L_{m+n}^+ + \text{ dim. of spacetime}$$

So need to find algebra of Viroso operators.