

8.251 – Homework 4

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Due Wednesday, March 7.

1. (10 points) Problem 5.6
2. (15 points) Point particle action in curved space.

(This problem is an elaboration of Problem 5.7 in the textbook.)

In Section 3.6 we considered the invariant interval $ds^2 = -g_{\mu\nu}(x)dx^\mu dx^\nu$ in a curved space with metric $g_{\mu\nu}(x)$. The motion of a point particle of mass m on curved space is studied using the action

$$S = -mc \int ds .$$

(a) Show that the equation of motion takes the form

$$\frac{d}{ds} \left[g_{\rho\mu} \frac{dx^\mu}{ds} \right] = \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\rho} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} .$$

This equation of motion is called the geodesic equation. Although the action is simple, the geodesic equation is nonetheless rather nontrivial. However, if the metric is constant, as in special relativity, then the right-hand side of the above equation vanishes, and we recover the familiar equation of motion of a free point particle.

(b) Another form of the geodesic equation can be found by expanding the derivative on the left-hand side, and then manipulating the result until one obtains an equation for d^2x^λ/ds^2 . Show that this equation can be written as

$$\frac{d^2x^\lambda}{ds^2} + \Gamma_{\mu\nu}^\lambda \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 ,$$

where

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\sigma\mu}}{\partial x^\nu} + \frac{\partial g_{\sigma\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right) .$$

The Christoffel coefficients Γ are symmetric in the lower indices $\Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda$, and they vanish if the metric is constant.

3. (10 points) Schwarz inequality in $\mathbb{R}^{1,1}$.

Consider a two-dimensional vector space V with a constant metric such that there is a spacelike vector s' ($s'^2 = s' \cdot s' > 0$) and a timelike vector t' ($t'^2 = t' \cdot t' < 0$). As usual, we write $v_1 \cdot v_2 = g_{ij} v_1^i v_2^j$, where g_{ij} are the components of the metric.

(a) Prove that you can find vectors s and t such that $s \cdot s = 1$, $t \cdot t = -1$ and $t \cdot s = 0$. The vectors t and s provide a canonical basis for V , which is now identified as the space $\mathbb{R}^{1,1}$. Namely, we can say $t = (1, 0)$, $s = (0, 1)$ together with $g_{00} = -1$, $g_{11} = 1$ and $g_{01} = g_{10} = 0$.

(b) Consider now two arbitrary vectors v_1 and v_2 in V . Prove that

$$(v_1 \cdot v_2)^2 \geq v_1^2 v_2^2,$$

where the equality only holds if the vectors v_1 and v_2 are parallel.

4. (10 points) Metric on S^2 from stereographic parameterization.

Consider a unit sphere S^2 in \mathbb{R}^3 centered at the origin: $x^2 + y^2 + z^2 = 1$. Denote a point on the sphere by $\vec{x} = (x, y, z)$. In the stereographic parameterization of the sphere we use parameters ξ^1 and ξ^2 and points on the sphere are (see (6.1)):

$$\vec{x}(\xi^1, \xi^2) = \left(x(\xi^1, \xi^2), y(\xi^1, \xi^2), z(\xi^1, \xi^2) \right).$$

Given parameters (ξ^1, ξ^2) , the corresponding point on the sphere is that which lies on the line that goes through the north pole $N = (0, 0, 1)$ and the point $(\xi^1, \xi^2, 0)$. Note that the north pole itself is not attained for any finite values of the parameters.

(a) Draw a picture showing the above construction. What are the required ranges for ξ^1 and ξ^2 if we wish to parameterize the full sphere (except for the north pole)?

(b) Calculate the functions $x(\xi^1, \xi^2)$, $y(\xi^1, \xi^2)$, and $z(\xi^1, \xi^2)$.

(c) Calculate the four components of the metric $g_{ij}(\xi)$ defined in (6.14). This is the metric on the sphere, described using the ξ parameters. The algebra here is a bit messy, although the result is quite simple – it is OK to use a symbolic manipulator, like MatLab, Maple, or Mathematica.

(d) Check your result by computing the area of the sphere using (6.17).

5. (10 points) Problem 6.2 (restated here for your convenience with added notation)

Examine the Nambu-Goto action (6.39) for a relativistic string with endpoints attached at $(0, \vec{0})$ and $(a, \vec{0})$. Consider the non-relativistic approximation where $|\vec{v}_\perp| \ll c$ and the oscillations are small (see (4.3), whose left-hand side should have an absolute value!).

You may denote by \vec{y} the collection of transverse coordinates X^2, \dots, X^d and write $\vec{y}(t, x)$, where x is the coordinate corresponding to X^1 .

Work in the static gauge. Moreover, parameterize the strings using $X^1 = x = a\sigma/\sigma_1$. This parameterization is allowed for small oscillations. In fact, it is allowed for any motion in which X^1 is an increasing function along the string.

Show that the action reduces, up to an additive constant, to the *action* for a non-relativistic string performing small transverse oscillations. What is the tension and the linear mass density of the resulting string? What is the additive constant?

6. (10 points) Problem 6.5.