MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.282 May 23, 2002

Final Exam

Name ________________________________ **(please print)**
Last **First**

1. Work all problems.
2. Closed book exam; you may use two pages of notes.
3. Wherever possible, try to solve the problems using general analytic expressions. Plug in numbers only as a last step.

Problem	Grade	Grader
1		
2		
3		
4		
5		
6		
7		
8		
Total		

USEFUL CONSTANTS

Constant	cgs units		mks units	
c (speed of light)	3×10^{10}	cm/sec	3×10^{8}	m/sec
G (gravitation constant)	7×10^{-8}	dyne-cm^2/g^2	7×10^{-11}	N-m^2/kg^2
k (Boltzmann's constant)	1.4×10^{-16}	erg/K	1.4×10^{-23}	J/K
h (Planck's constant)	6.6×10^{-27}	erg-sec	6.6×10^{-34}	J-sec
m_{proton}	1.6×10^{-24}	g	1.6×10^{-27}	kg
eV (electron Volt)	1.6×10^{-12}	erg	1.6×10^{-19}	J
$M_\odot$ (solar mass)	2×10^{33}	g	2×10^{30}	kg
$L_\odot$ (solar luminosity)	4×10^{33}	erg/sec	4×10^{26}	J/sec
$R_\odot$ (solar radius)	7×10^{10}	cm	7×10^{8}	m
σ (Stefan-Boltzmann cons)	6×10^{-5}	erg/cm^2-sec-K^4	6×10^{-8}	J/m^2-sec-K^4
Å(Angstrom)	10^{-8}	cm	10^{-10}	m
km (kilometer)	10^{5}	cm	10^{3}	m
pc (parsec)	3×10^{18}	cm	3×10^{16}	m
kpc (kiloparsec)	3×10^{21}	cm	3×10^{19}	m
Mpc (megaparsec)	3×10^{24}	cm	3×10^{22}	m
year	3×10^{7}	sec	3×10^{7}	sec
day	86400	sec	86400	sec
AU	1.5×10^{13}	cm	1.5×10^{11}	m
$1'$ (arc minute)	1/3400	rad	1/3400	rad
$1''$ (arc second)	1/200,000	rad	1/200,000	rad

Problem 1 (13 points)

a. A neutron star whose mass is 1.4 $M_\odot$ is in orbit about a normal star. Matter is flowing from the normal star onto the collapsed star at a rate of 10^{17} grams per second.

i. Assume that all of the gravitational potential energy of the infalling matter is converted to radiation. Compute the bolometric luminosity of the neutron star.

ii. Suppose that the surface of the neutron star radiates this energy as a blackbody of a single temperature T. Compute T. Also compute the quantity kT (in eV) and comment on the type of electromagnetic radiation you think is being emitted. (Take the radius of the neutron star to be $R_{\rm NS} = 10$ km.)

$$L = \frac{GM\dot{M}}{R} \quad \text{(a factor of 1/2 is optional)}$$

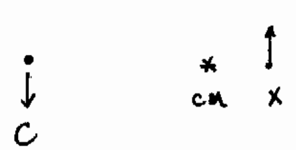
$$= 1.9 \times 10^{37} \text{ ergs/sec}$$

$$\sigma T^4 4\pi R^2 = L \Rightarrow T_{eff} = \left(\frac{L}{4\pi R^2 \sigma}\right)^{1/4}$$

$$T_{eff} \simeq 12.7 \times 10^6 \text{ K}$$

$$kT \simeq 1.75 \times 10^{-9} \text{ ergs} \cong 1 \text{ keV} \quad \text{(X-ray band)}$$

b. A new X-ray pulsar was discovered by MIT scientists last week. It has an orbital period of 42 minutes. The neutron star moves in a circular orbit about the *center of mass* of the binary with a velocity of 11 km s^{-1}, while its unseen companion star has an orbital velocity of 770 km s^{-1}. Find the masses of the unseen companion star and the neutron star (in units of $M_\odot$).



$$a_c = \frac{v_c P}{2\pi} = \frac{770 \times 42 \times 60}{2\pi} = 309{,}000 \text{ km}$$

$$a_x = \frac{v_x P}{2\pi} = 4400 \text{ km}$$

Kepler's 3rd law: $\dfrac{GM_T}{a^3} = \left(\dfrac{2\pi}{P}\right)^2 \Rightarrow M_T = \left(\dfrac{2\pi}{P}\right)^2 \dfrac{a^3}{G} = 2.87 \times 10^{33}$ grams

$$M_T = 1.435\, M_\odot$$

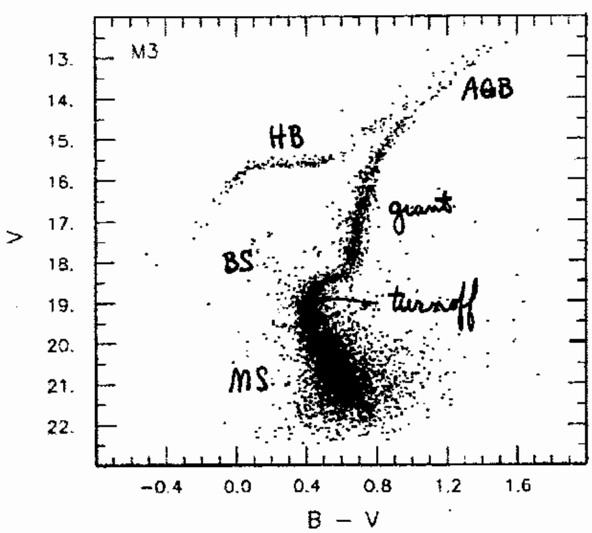
$$(M_c + M_x) = M_c\left(1 + \frac{M_x}{M_c}\right) = 1.435$$

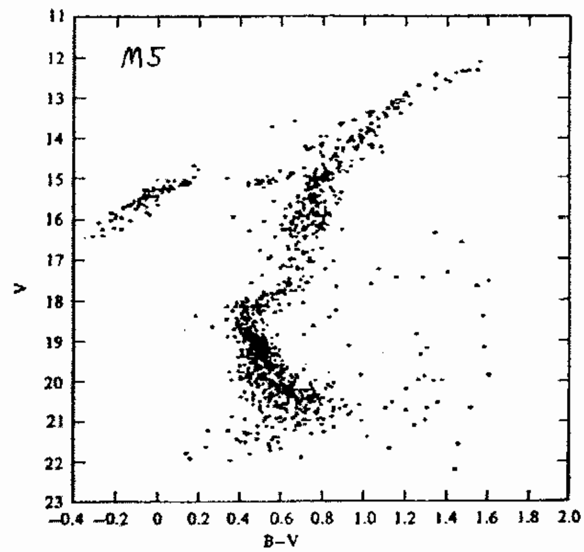
$$M_c\left(1 + \frac{M_x}{M_c}\right) = M_c\left(1 + \frac{a_c}{a_x}\right) = M_c\left(1 + \frac{770}{11}\right) = 1.43\, M_\odot$$

$$M_c \simeq 0.02\, M_\odot$$

Problem 2 (12 points)

The figures below show H–R diagrams for two globular clusters: M3 and M5. (V is the apparent visual magnitude.)





a. Identify the various stellar evolutionary phases seen on the H–R diagram for M3. You may write your answers directly on the plot.

b. Find the relative distances to the two clusters, i.e., d_{M3}/d_{M5}.

c. What are the predominant nuclear reactions powering the different evolutionary phases, and where are they taking place within the star (e.g., in a shell or in the core)?

d. How many main-sequence stars with $V = 20$ equal the brightness of a single giant with $V = 12.5$?

b. The horizontal branch for M3 is at ~ $V = 15.65$, that for M5 at ~ 15.25. Thus, M5 has a distance modulus that is ~ 0.4 magnitudes greater than that for M3. Each 5 magnitudes is a factor of 10 farther away.

Therefore $\frac{d_{M5}}{d_{M3}} = 10^{0.4/5} = 10^{0.08} = 1.2$ (or $d_{M3}/d_{M5} = 0.83$)

c. On the main sequence H → He in the core

Giant branch H → He in a shell around the core of He

AGB H → He and He → C in shells around the CO core

Horizontal branch → He → C in the core

d. $V = -2.5 \log\left(\frac{f_{MS}}{f_0}\right) = 20$

Require $12.5 = -2.5 \log\left(\frac{N f_{MS}}{f_0}\right) = -2.5 \log N - 2.5 \log\left(\frac{f_{MS}}{f_0}\right)$

$12.5 = -2.5 \log N + 20$

$\log N = -7.5/-2.5 = 3$ $\qquad N = 1000$

Problem 3 (Short Answer Questions – 13 points)

a. Use a dimensional analysis of the equation of hydrostatic equilibrium to derive a radius–mass relation, $M(R)$, for white dwarfs. Assume that the equation of state for degenerate matter is $P = K\rho^{5/3}$, where P is the pressure, ρ is the mass density, and K is a constant.

$$\frac{dP}{dr} = -g\rho \qquad P = K\rho^{5/3}$$

Dimensional analysis $\frac{P}{R} \simeq \left(\frac{GM}{R^2}\right)\left(\frac{M}{R^3}\right) \simeq \frac{K\rho^{5/3}}{R} \simeq K\left(\frac{M}{R^3}\right)^{5/3}\frac{1}{R}$

or $\frac{GM^2}{R^5} \sim \frac{KM^{5/3}}{R^6}$

$$R \sim \frac{K}{G} M^{-1/3}$$

b. What is 21-cm radiation, and what can be learned by studying it?

(Hyperfine) Interaction between the proton and electron magnetic moments in hydrogen. The 21-cm line results from a "spin flip" (transition from F=1 to F=0 state). It is used to map the density and velocity of hydrogen (neutral) in our galaxy and in neighboring spiral galaxies.

c. What are the basic ingredients required to compute the "Oort limit" on the local mass density in the galactic plane? (Not to be confused with the Oort A and B coefficients for studying galactic rotation.)

Near the galactic plane (but outside most of the matter density associated with the disk) gravity, g, is approximately constant. If test stars have an average KE $\sim \langle \frac{1}{2} m_* v_*^2 \rangle$, then their scale height above the disk is

$$H \sim \frac{\langle \frac{1}{2} m_* v_*^2 \rangle}{m_* g} \qquad \text{Thus, } g \simeq \frac{\langle v_*^2 \rangle}{2H}$$

So, we need information about the stellar scale height and velocity dispersion.

Problem 4 (Short Answer Questions – 12 points)

a. A globular cluster contains 10^6 stars. Assume they all have the same mass, m and the same speed, v. Use the *virial theorem* to estimate a characteristic size of the cluster.

virial theorem: $\langle KE \rangle = -\frac{1}{2} \langle PE \rangle$

$$N \frac{m v^2}{2} \sim \frac{1}{2} \frac{G (N m)^2}{R}$$ ← characteristic radius

$$R \sim \frac{G N m}{v^2} = \frac{10^6 G m}{v^2}$$

if $v \approx 10$ km/sec and $m \approx 1 M_\odot$, $R \sim 40$ pc

b. Explain what the cosmic microwave background (CMB) is, and how it originates.

The CMB is the "cooled" blackbody radiation left over from the time when hydrogen became neutral (recombination) at a z of ~ 1000. The radiation is now at $T \simeq 2.76$ K and was at ~ 3000 K at recombination.

c. Name three effects that dust has on starlight propagating through it.

reddens the light (by preferentially scattering away the blue)
dims the light (i.e., increases its magnitude) by extinction
polarizes it (at the $\sim 1\%$ level).

Problem 5 (13 points)

a. A reasonable approximation to the $M_V - P$ relation for Cepheid variables is:

$$\langle M_V \rangle \simeq -2.3 \log_{10}(P/\text{days}) - 1.7$$

A Cepheid variable with mean apparent magnitude $\langle m \rangle = 19$ is found in the galaxy M31 with a pulsation period of 30 days. Find the distance to M31 in terms of a distance modulus and in parsecs.

The relation yields $\langle M_V \rangle = -5.1$

$\langle m \rangle \equiv \langle V \rangle = 19$

Thus, the distance modulus $= 24.1$

each 5 magnitudes of which is another factor of 10 in distance

Thus, $d \simeq 660{,}000$ pc

b. There are a number of different "standard candles" that are very useful in astronomy for estimating distances to *extragalactic* objects. Name and briefly discuss 3 of these (not including Cepheids).

	M_V
Type I a SNe	-18
Globular clusters	-9
Brightest galaxy in a cluster	-21
Planetary nebulae	-6
HII regions	-9
Novae	-8

absolute magnitudes

Problem 6 (12 points)

A spiral galaxy is observed to have a rotation curve that can be well fitted by the following analytic expression:

$$v(R) = 200\left(1 - e^{-R/R_0}\right)\ \mathrm{km/sec}\,,$$

where e^{-R/R_0} is an exponential function of R, the radial distance from the center of the galaxy, and R_0 is a constant whose value is 4 kpc. (Assume that the galaxy is viewed edge on so that the maximum Doppler velocities are observed.)

a. Estimate how much mass there is interior to a distance of 16 kpc from the center. (Express your answer in units of $M_\odot$.)

b. Show that for small distances from the galaxy center ($R \ll 4$ kpc) the angular frequency, Ω, is approximately constant. [Hint: Use a Taylor's series expansion for $v(R)$.] (If you are not able to do this part analytically, you may demonstrate the answer numerically with your calculator.)

c. Find the rotation period ($2\pi/\Omega$) of stars near the center, in units of years.

d. For large distances, R, the rotation curve approaches a nearly constant value. Assume that the gravitating matter which produces this rotation curve is distributed spherically symmetrically about the center with a mass density that falls off with distance as $\rho(R) \propto R^{-\alpha}$, for $R \gg 4$ kpc. Find the value of α that leads to a flat rotation curve. You should consider only large distances in this part of the problem.

a. At 16 kpc $\quad v(R) = 200\left(1 - e^{-16/4}\right) \simeq 200$ km/sec

$$\frac{GM(<R)}{R^2} \sim \frac{v^2}{R} \qquad M(<R) \simeq \frac{v^2 R}{G} \simeq 2.9\times10^{44}\,\mathrm{g} \simeq 10^{11}\ M_\odot$$

b. $$v(R) \simeq 200\left(1 - 1 + R/R_0\right) \simeq \frac{200\,R}{R_0} \quad \text{for } R \ll R_0$$

but $v = \Omega R \propto \dfrac{200\,R}{R_0}$ $\qquad \Omega \simeq 200\ \mathrm{km/sec}/4\ \mathrm{kpc}$

c. $$P = \frac{2\pi}{\Omega} = \frac{2\pi}{200}\ \frac{4\ \mathrm{kpc\ sec}}{\mathrm{km}} = 0.126 * 3\times10^{21}/10^5 = 3.8\times10^{15}\ \mathrm{sec}$$

$P \sim 120$ million years

d. from parts (a) $\quad \dfrac{GM(<R)}{R^2} \sim \dfrac{v^2}{R} \qquad M(<r) \sim \dfrac{v^2 R}{G}$

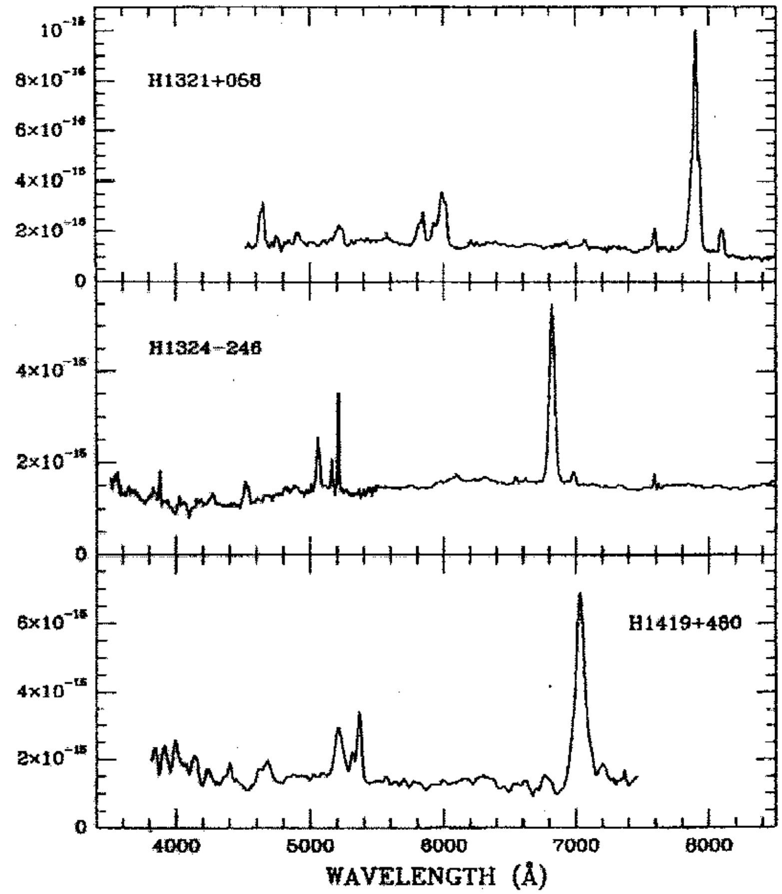
if $\rho \propto R^{-\alpha}$ then $M(<R) \propto R^{-\alpha+3}$ (volume $\propto R^3$)

$$R^{3-\alpha} \propto \frac{v^2 R}{G} \quad \Rightarrow \quad \underline{\alpha = 2}$$

or $\rho \propto R^{-2}$

Problem 7 (12 points)

The plots below show three quasar spectra. The most prominent line in each spectrum is due to Hα with a rest wavelength of 6562 Å. For the purpose of this problem, assume a value for the Hubble constant, H_0, of 70 km s^{-1} Mpc^{-1}.



a. For the *most distant* of these quasars, find its approximate red shift, $z = (\lambda - \lambda_0)/\lambda_0$.
b. What is the approximate recession velocity of this quasar?
c. Estimate the distance to the quasar.
d. The energy flux from this quasar is 3×10^{-13} ergs cm^{-2} sec^{-1} (3×10^{-16} Watts m^{-2}), a substantial fraction of which is emitted in the Hα line. Find the approximate luminosity of the quasar.
e. A model for the central object in this quasar involves a supermassive black hole of $10^7\ M_\odot$. Is this consistent or not with the concept of an "Eddington limit"? Explain.

a. $\lambda \simeq 7900$ Å $z = (7900 - 6560)/6560 = 0.204$

b. $z \approx v/c$ $v \approx 0.2c \approx 60{,}000$ km/sec

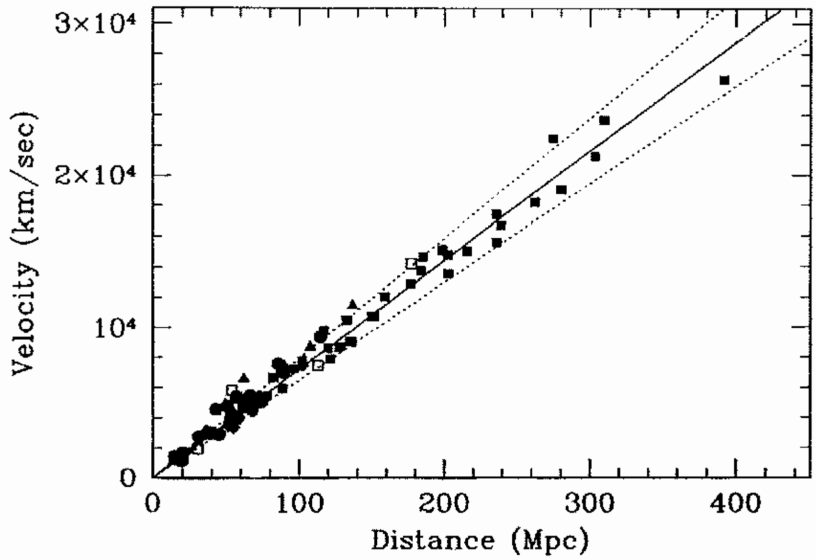
c. $v = Hd$ $d \simeq 60{,}000$ km/sec$/H \simeq 60{,}000/70 \simeq 860$ Mpc

d. $L \simeq 4\pi d^2 f = 12\,(860\ \text{Mpc})^2\ 3\times10^{-13} \simeq 2.6\times10^{43}$ ergs/sec $\simeq 2.6\times10^{36}$ Watts

e. $L_{Edd} \simeq 10^{38} \left(\frac{M}{M_\odot}\right)$ ergs/sec $\simeq 10^{45}$ ergs/sec for a $10^7\ M_\odot$ black hole
since $L \ll L_{Edd}$, this is consistent with the supermassive BH model

Problem 8 (13 points)

The graph below shows a plot of the recessional velocity of galaxies as a function of their distance.



a. Use the diagram to estimate the Hubble constant, H_0 (to better than a factor of 2).
b. Use the answer to part (a) to estimate the age of the universe.
c. For the case of a "flat", $\Omega_0 = \Omega_m = 1$ universe, use Newtonian cosmology to derive a more exact relation between the Hubble constant and the current age of the universe.

a. $H_0 = v/d = \frac{\Delta v}{\Delta d} = \frac{3\times 10^4 \text{ km/sec}}{400 \text{ Mpc}} \simeq 75 \text{ km/sec/Mpc}$

b. $t_0 \equiv \frac{1}{H_0} = \frac{\text{Mpc}}{75 \text{ km/sec}} = \frac{3\times 10^{24} \text{ cm sec}}{75 \times 10^5 \text{ cm}} \simeq 4\times 10^{17} \text{ sec}$

$t_0 \simeq 12.7 \text{ Gyr}$

c. Use either $\frac{1}{H_0^2}\left(\frac{da}{dt}\right)^2 - \frac{\Omega_m}{a} = 0$ where a = scale factor

or its solution $a = (\frac{3}{2} H_0 t)^{2/3}$

At the current epoch $a = 1$; the big bang corresponds to $a = 0$

$1 = \frac{3}{2} H_0 t$

$t = \text{age} = \frac{2}{3}\frac{1}{H_0}$