

- Reading: Binney and Tremaine: Chapter 1 and §3.1.
- Note: This set requires numerical integration of differential equations and probably cannot be finished if started the night before it's due!

(1) The potential for the Mestel disk is given by

$$\Phi(r, \theta) = v_c^2 \left(\ln \frac{r}{r_0} + \ln \frac{1 + |\cos \theta|}{2} \right) .$$

- Using Gauss' law, verify that the axisymmetric thin disk giving rise to the Mestel potential, has a surface density μ which varies as $1/r$.
 - What is the surface density in terms of v_c and r ? What would the surface density (in M_\odot/pc^2) be at the Sun's position in the Milky Way, where $v_c = 220$ km/s and $R = 8$ kpc, if the Milky Way had a Mestel disk potential?
- (2) A parabolic telescope mirror, the primary, has diameter D and focal length L , giving it an f -ratio of L/D . Suppose that a hyperbolic secondary is installed a distance $L/3$ from the prime focus bringing on-axis light to a perfect focus at the center of primary mirror (we can cut a hole in the primary since its center will be occulted by the secondary). This is a *Cassegrain* configuration. The prime and secondary foci of the telescope are the foci of the hyperbola.
- Why is the secondary hyperbolic, rather than spherical, elliptical, parabolic, or for that matter anything else?
 - What is the f -ratio seen at the secondary focus. What is the *effective* focal length of the telescope?
- (3-5) Consider a star orbiting in the spherically symmetric gravitational potential $\Phi = v_c^2 \ln(r/r_0)$.
- Use conservation of energy and angular momentum to obtain equations for dr/dt and $d\theta/dt$.
 - Take the radius and velocity at apogalacticon to be r_a and v_a . Use these two dimensioned quantities and v_c to recast your answer in a) in terms of dimensionless quantities (e.g. you might let $\xi = r/r_a$ and $\zeta = v_c t/r_a$).
 - You now have two coupled first order differential equations for the orbit. You might be tempted to use your Runge-Kutta integrator to compute ξ and θ as functions of ζ . *Don't!* You would find that one of the first derivatives had a sign ambiguity which would be tricky to resolve in your program. Instead, use brute force. Turn your first order equation for ξ into a second order equation by differentiating with respect to ζ . (If you've taken 8.033 you might have taken an alternate route to these equations by constructing the Lagrangian and writing down Lagrange's equations.)
 - In parts a-c) you reduced the problem of the orbit two coupled differential equa-

tions,

$$\xi'' = \frac{k^2}{\xi^3} - \frac{1}{\xi} \quad \text{and} \quad \theta' = \frac{k}{\xi^2} \quad ,$$

where the prime denotes differentiation with respect to ζ and $k = v_a/v_c$. Use your Runge-Kutta integrator to integrate these equations for an orbit which is at apocenter at “time” $\zeta = 0$ with $v_a/v_c = 0.451$. You will want to turn the second order differential equation into two coupled first order equations for a total of 3 coupled first order equations. Show your code.

- d) Plot the above orbit for 20π radians. Use discrete points for the first orbit and a continuous line for the remainder.
- e) By what angle does the orbit advance in the going from one apogalacticon to the next? What is the ratio of apogalacticon to perigalacticon for this orbit?

The ratio of $v_a/v_c = 0.451$ was chosen with malice aforethought. An orbit with $v_a/v_c = 0$ would have been linear with all its energy in the radial direction. An orbit with $v_a/v_c = 1$ would be circular, with all its energy in the tangential direction. Our orbit was chosen so that a superposition of such orbits at random orientations and random phases with varying scale lengths would look like an round galaxy with equal energy in the radial and each of the two tangential directions, i.e., like an isothermal gas. But note that since the stars don't collide, the orbit plunges through a large range of radii!