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PROFESSOR: A quick review of what we talked about last time. So the first thing we did last time was to discuss the age of universe, considering so far only at this point, flat matter-dominated universes where the scale factor goes like t to the $2/3$. And we were easily able to see that the age of such a universe was $2/3$ times h inverse. And we did discuss what happens if you plug-in numbers into that formula using the best current value of h , the value obtained by the Planck team last March. The age turns out to be about 9.5 to 9.9 billion years. And that can't be the real age of the universe, we think, because there are stars that are older than that. The age of the stars seems to indicate that the universe should be at least, according to one paper I cited last time, 11.2 billion years old, and this is younger.

So the conclusion is that our universe is not a flat, moderate, matter-dominated universe. We do in fact have good evidence that the universe is very nearly flat, so it's the matter-dominated part that has to fail, and it does fail. We also have good evidence that the universe is dominated today by dark energy, which we'll be talking about later. But one of the pieces of evidence for this dark energy is this age calculation. The age calculation just does not work unless you assume that the universe has a significant component of this dark energy, which we'll be discussing later.

We then talked about the big bang singularity, which is an important part of understanding, when you talk about the age, what exactly you mean by the age, age since when. And the point that I tried to make there is that the big bang singularity, which gives us mathematically the statement that the scale factor at some time which we call zero was equal to zero, and if you put back that formula into other formulas, you discover that the mass density, for example, was infinite at this magical time that we call zero. That singularity is certainly part of our

mathematical model and doesn't go away even when we make changes in the mathematical model.

But we don't really know if it's an actual feature of our universe, because there's certainly no reason to trust this mathematical model all the way back to t equals zero, where the densities become infinite. We know a lot about matter and we think we can predict how matter will behave, even the temperatures and energy densities somewhat beyond what we measure in laboratory. But we don't think we can't necessarily extrapolate all the way to infinite matter density. So these equations do break down when you get very close to t equals zero and nobody really knows exactly what should be said about t equals zero. When we talk about the age, we're really talking about the age since the extrapolated time at which a would have been zero in this model, but we don't really know that it ever actually was zero.

We next discussed the concept of the horizon distance. If the universe, at least the universe as we know it, we want to be agnostic about what happened before t equals zero, but certainly the universe as we know it, really began at t equals zero in the sense that that's when structure and complicated things started to develop. So since t equals zero there's been only a finite amount of time elapsed. And since light travels at a finite speed, that means that light could only have traveled some finite distance since the big bang, since t equals zero. And that means that there's some object which is the furthest possible object that we could see, and any object further than that would be in a situation where light from that object would not yet have had time to reach us. And that leads to this notion of our horizon distance, where the definition of the horizon distance is that it is defined to be the present distance of the most distant objects which we are capable of seeing, limited only by the speed of light. And once able to calculate that, and in particular for the model that we so far understood at this point, the matter-dominated flat universe, the horizon distance turned out to be three times $c t$. Tt

Now remember, if the universe were just static and appeared at time t ago, then the horizon distance would just be $c t$, the distance that light could travel in time t . What makes it larger is the fact that the universe is expanding, and that means that

everything was closer together in the early time and light could make more progress at the early time, and then these objects have since moved out to much larger distances. So that allows the horizon distance to be larger than $c t$, and in this particular model, it's three times $c t$.

Next we began a calculation, which is what we're going to pick up to continue on now. We were calculating how to extend our understanding of a of t , the behavior of the scale factor, away from the flat case, to ultimately discuss the two other cases, the open and closed universe. And we decided on a flip of a coin-- I promise you I flipped a coin at some point-- to start with the closed universe. We could have done either one first. The equation that we start with is basically the same, it's the sine of k that makes the difference. This is the so-called Friedmann equation, and for a closed universe k is positive. This is the evolution equation, but it has to be coupled with an equation that describes how ρ behaves with time. And for a matter-dominated universe, ρ is just representing non-relativistic matter, which is nothing but spread as the universe expands. And the spreading gives a factor of one over a cubed as the volume grows as a cubed. And that means that ρ times a cubed is a constant, and that expresses everything that there is to know about how ρ behaves with time.

OK, then after writing these equations, we said that things will simplify a little bit, not a lot, but a little bit if we redefine variables basically to incorporate all the constants that appear in these equations into one overall constant. And we decided, or I claimed, that a good way to do that, an economical way to do that, is to define things so that the variables all have units which are easily understood. And in this case the units of length can describe everything that we need, so we chose to express everything in terms of variables that have units of length.

So the scale factor itself is units of meters per notch, and that's not a length. And notches we'd like to get rid of because we know they're un-physical. That is, there's no standard for what the notch should be. So if we divide a of t by the square root of k , the notches disappear, and we get something which just has the units of meters, or units of length, and I call that a twiddle of t . Similarly, but more obviously, t can be

turned into a length by just multiplying by c , the speed of light. So I defined a variable \tilde{t} which is just c times t . So both of these new variables, with the tildes, have units of length.

And then the Friedmann equation can be rewritten, just reshuffling things according to these new definitions, in this way where all the constants are lumped into this variable α , where α is this complicated expression, which absorbs all the constants from the earlier equation. α also has units of length, and even though it has a ρ times \tilde{a} cubed in it, and both ρ and \tilde{a} each depend on time, the product of ρ times \tilde{a} cubed is a constant because ρ times \tilde{a} cubed is a constant and \tilde{a} differs only by the square root of k , which is also a constant. So α is a constant even though it has time dependent factors. The time dependence of those two factors cancel each other out to give something which is time independent and can be evaluated any old time.

So this now is our equation, and we proceeded to manipulate it. So the first thing we did was to take it's square root, and to rearrange it so that $d\tilde{t}$ appeared on one side, and everything else was on the other side. And everything else depends on \tilde{a} but not explicitly on time. So this completely separates everything that depends on \tilde{t} on the left, and everything that depends on \tilde{a} on the right. And now we can just integrate both sides of that equation, and and they point to that, that there are basically two ways of proceeding here, one of which we already did when we did the flat case. When we did the flat case, we integrated both sides as indefinite integrals. And when you carry out an indefinite integration, you get a constant of integration, which then becomes a constant in your solution. And in that case we discovered that the constant really just shifted the origin of time. And since we had not said anything previously that in any way determined the origin of time, we used that constant to arrange the origin of time so that \tilde{a} of zero would equal zero, and that eliminated the constant.

Just for variety, I am going to do it another way this time. Instead of doing an indefinite integral, I will do a definite integral. And if you do a definite integral, you have to make sure you're integrating both sides over the same range, or at least

corresponding ranges. We have different names for the variables, but the range of integration of the two sides has to match in order to maintain the equality between the two sides.

So we're going to integrate the left side from zero to some final time. And I'm going to call the final time t tilde variable sub f , for f to stand for final. And there's no real final time here. It could be any time, it's just the final time for the integration, and in the end discover then how things behave at time t tilde sub f . And then once we've figured that out we can drop the effort. It will be a formula that will be valid for any time.

So the left hand side is integrated from zero to time t tilde f , the right hand side has to be integrated over the corresponding time interval. And we would like to define a t tilde and the origin of time so that a tilde equals zero at time zero, the same convention we used for the other case. The standard convention in cosmology, t equals zero, is the instance of the Big Bang. And the instance of the Big Bang is the time at which the scale factor vanished. So that means when we have a lower limit of integration of t tilde equals zero on the left hand side, we should have a lower limit of integration of a tilde equals zero on the right hand side.

And similarly the upper limits of integration should correspond to each other. So the upper limit of integration on the left hand side was t tilde sub f , really just an arbitrary time that we designated by the subscript f . So the right hand side the limit should be the value of a tilde at that time. And I'll call that a tilde sub f . So a tilde sub f is just defined to be the value of a tilde at the time t tilde sub f . And in this way the limits of integration on the two sides correspond and now we can integrate them and we don't need any new integration constants. These definite integrals have definite values, which is why they're called definite integrals, I suppose.

OK any questions about that, because that's where we're ready to take off and start doing new material. Yes.

AUDIENCE:

I have a general question. So regarding the issue of how we're not really sure how to extrapolate up to t equals zero, this is just kind of a general question. I was

wondering how we're kind of assuming throughout that time kind of flows uniformly, and so I've heard about something like gravitational time dilation. So at the beginning especially when there's such a high density of matter or radiation, then wouldn't that affect how, I guess, time flows?

PROFESSOR: OK good question, good question. The question was, does things like general realistic time dilation affect how time flows, and are we perhaps being overly simplistic and assuming that time just flows smoothly from time zero onward. The answer to that is that general relativity does predict an extra time dilation, which in fact is built into the Doppler shift calculation that we already did. And there are other instances where similar things happen. If a photon travels from the floor of this room to the ceiling of this room, there's a small Doppler shift, a small shift in the timing. And you could see it in principle with clocks as well. If you had a clock on the floor, and a clock on the ceiling, they would not run at quite the same rate. But to talk about time dilation, you always have to have two clocks to compare. In the case of the universe, we have this thing that we call cosmic time, which can be measured on any clock. The homogeneity assumption implies that all clocks will do the same thing, so so the issue of time dilation really does not arise. Our definition of cosmic time defines time that is the time variable that we will use. And when we say it flows uniformly from zero up to the present time, that word uniformly sounds sensible. But if you think about it, I don't know what it means. So I don't even know how to ask if it's really uniform or not. It's certainly true that our time variable evolves from zero up to some final time, but smoothly or uniformly is not really a question we can ask until one has some other clock that one can compare it to.

PROFESSOR: Yes.

AUDIENCE: So you said at the beginning that something like an infinite density is just an effect of our equation. Aren't a lot of theories like inflation come out of assuming that the universe had gotten some infinite, dense, small area? So what effect does this assumption that may or may not be correct have on theories?

PROFESSOR: OK, the question is, if we are not sure we should believe the t equals zero

singularity, how does that affect other theories like inflation, which are based on extrapolating backwards to very early times. And there is an answer to that. The answer may or may not sound sensible, and it may or may not be sensible. It's hard to know for sure. But one can be more detailed and ask how far back do we think we can trust our equations? And nobody really knows the answer to that of course, that's part of the uncertainty here. But a plausible answer which is kind of the working hypothesis for many of us, is that the only obstacle to extrapolating backwards is our lack of knowledge of the quantum effects of gravity, and therefore the quantum effects of what space time looks like. And we can estimate where that sets in. And it's at a time called the Planck time, which is about 10^{-43} seconds, and inflation, which we'll see later as sort of a natural timescale of about probably 10^{-37} seconds. So although that's incredibly small, it's actually incredibly large compared to 10^{-43} seconds. So we think there is at least a basis for believing that things like our discussions of inflation which we'll be talking about later, are valid even though we don't think we can extrapolate back to $t = 0$. Yes.

AUDIENCE: I have a question about use of the definite integrals.

PROFESSOR: Yes.

AUDIENCE: So we have a twiddle defined as a over square root of k . And we noticed in our equation that a goes to zero as t goes to zero, so a twiddle also goes to zero. How do we know then that that definite integral is convergent, because then we have zero over zero competing case. [INAUDIBLE]

PROFESSOR: Let me think. Yeah, we'll see, I think is the answer. How do we know it's convergent? Well, we're going to actually do the integral, and then the integral does converge. You are right. The integrand does become zero over zero, but that in fact means the integrand is some finite number actually. Both numerator and denominator go to zero, I mean let me think about this. I guess a tilde squared becomes negligible, so the denominator goes like one over the square root of a tilde. So in fact the numerator divided by the denominator goes like the square root

of a tilde as time goes to zero. Because you have a square root in the denominator, the a tilde squared becomes negligible, so it's manifestly convergent. The integrand does not even blow up, even though a tilde does go to zero. It's certainly worth looking at, you're right. One should always check to make sure integrals are convergent. But since we will actually be explicitly doing the interval, if it was divergent we would get a divergent answer, and we will not as you'll see in a couple minutes.

Any other questions? OK, so in that case, to the blackboard.

OK, so I will write on the blackboard the same equation we have up there, so we can continue. $t \ddot{f}$ is equal to the integral from zero to $a \ddot{f}$, $a \ddot{d} \ddot{f}$ over the square root of $2\alpha a \ddot{f} - a \ddot{f}^2$.

OK, so what we'd like to do now is to carry out the integral on the right hand side. Ideally what we'd like to do is to carry out the integral on the right hand side and get some function of a tilde f and then we'd like to convert that function to be able to write a tilde f as a function of time. We actually won't quite be able to do that. We'll end up with what's called the parametric solution, and you'll see how that arises and what that means. I don't need to try to describe exactly in advance what that means.

Doing the integral can be done by some tricks, some substitutions. And the first substitution is based on completing the square in the denominator, and that motivates the substitution that we will make. So we can rewrite this just by doing some algebra on the denominator, which is called completing the square for reasons that you'll see when I write down what it is. The numerator will stay a tilde d a tilde. And the denominator can be written as $\alpha^2 - a \ddot{f} - \alpha$ quantity squared. So completing the square just means to put the a tilde inside a perfect square. And the nice thing about this is that now we can shift our variable of integration, and turn this into just a single variable instead of the sum of the two. And then you have an expression, which is clearly simpler looking than this one, which had a tildes in both places. Now the variable integration will appear only

there. And substitution which does that obviously enough, is we're going to let something, we can choose whatever we want, and then once I call it x , we're going to let x equal $a - \alpha$. And we're just going to rewrite that integral in terms of x .

So what we get when we do that is $\int f(a) dx$ is equal to the rewriting of that integral, and just substituting the a becomes $x + \alpha$. So we get $x + \alpha$ where a was, and da becomes just dx . And the denominator, which was our motivation for making the substitution in the first place, becomes just $\alpha^2 - x^2$. So this is perfectly straight forward.

The next step which is important, is to get the limits of integration right, because with this definite integral method we really have to make sure that our limits of integration are correct. So straightforward to do that, the lower limit of integration was $a = 0$, and if $a = 0$, x is equal to $-\alpha$. So the lower limit of integration expressed as a value of x is $-\alpha$. And the upper limit expressed as x was $a = f$, and that means x is equal to $f - \alpha$. So the limit here is $f - \alpha$ for the upper limit of integration.

OK, now this integral is still not easy, but it can be made easy by one more substitution. And the one more substitution is a trigonometric substitution to simplify the denominator. We can let x equal $-\alpha \cos \theta$, where θ is our new variable of integration. And then $\alpha^2 - x^2$ becomes $\alpha^2 - \alpha^2 \cos^2 \theta$.

And the α s factor out and you have the square root of $1 - \cos^2 \theta$. $1 - \cos^2 \theta$ is $\sin^2 \theta$, which is a convenient thing to take the square root of, you just get $\sin \theta$. And everything else also simplifies. And the bottom line, which I will just give you, is that now we find that $\int f(x) dx$ is just the integral of $\frac{1 - \cos \theta}{\sin \theta} d\theta$. That's it. Everything simplifies to that which is easy to integrate.

We also have to keep track of our limits of integration. The limit of integration, the lower limit of integration, is where x equals $-\alpha$. And if x equals $-\alpha$

alpha, that means cosine theta equals 1. And cosine theta equals 1 means theta equals 0. So the limit of integration on theta is easy, it starts at 0. And the final value is really just a value that corresponds to the final value of x which is a twiddle f minus alpha. For now I'm just going to call it theta sub f. Things that we have just determined is the value of theta that goes with the upper limit there. We'll figure out in a second how to write it more explicitly.

Let me first do the integral to just get that out of the way. Doing the integral, you get alpha times 1 minus cosine theta sub f. So this in fact becomes half of our solution. It expresses t sub f in terms of theta sub f. And now that we're done with the whole problem, I'm going to drop the subscript f. There's just some time that we're interested in which is called t and the value of a at that time will be called a, and the value of theta at that time will be called theta. So I'm just going to drop the subscript everywhere because we are now in a situation where the subscript is everywhere, so dropping everywhere does not lose any information.

So one of our equations is going to be simply t is equal to alpha times 1 minus cosine theta. And another equation will come from figuring out what theta sub f really is, which I said comes from making sure that the upper limit of integration here corresponds to the upper limit of integration in the previous integral. So theta sub f, I had to determine this on another blackboard and then put the final equations together.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yes, that's right I don't want to drop twiddles. T twiddle, thank you, is equal to alpha times 1 minus cosine theta, and then we have the final value of x. Xx x is equal to a tilde minus alpha. So the final value of x is equal to the final value of a tilde minus alpha. And the final value of x is also related to theta by this equation, which is what we're trying to get, how theta is related to the other variables. So this is equal to minus alpha cosine of theta sub f. And now we might want to, for example, solve this for a tilde sub f, which just involves looking at the right hand half of the equation, bringing this alpha to the other side making a plus alpha. So this implies

that a tilde sub f is equal to alpha times 1 minus cosine theta sub f. So this equation now just says that theta sub f means what it should mean to give us a final limit of integration that corresponds to the final limit of integration on our original integral, which we called a tilde sub f. Yes.

AUDIENCE: Sorry, perhaps I missed it. But when you are doing the integral of 1 minus cosine theta, how come--

PROFESSOR: Oh, I got it wrong. Good point. The integral of cosine theta d theta is sine theta, not cosine theta.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Wait a minute. Hold on. Do I still have it wrong?

AUDIENCE: [INAUDIBLE]

PROFESSOR: OK hold on. There was an alpha missing here. That's part of the problem, coming from the original equation here. Let's see if I have this right now. This is copying from a long line altogether. So I got both factors wrong. And then wait a minute. I think this is right now. There's still a wrong [INAUDIBLE]?

AUDIENCE: [INAUDIBLE]

PROFESSOR: If I differentiate sine, I get cosine. So if I differentiate minus sine, I get minus cosine. So differentiating this, I get this. That should mean that integrating this I get that. I think I have it right. Sometimes I get things right. It's a surprise, but--

AUDIENCE: [INAUDIBLE]

PROFESSOR: What?

AUDIENCE: The last equation you wrote.

PROFESSOR: The last equation needs to be changed, right. This was copied from that, right. Thanks for reminding me. So t tilde is equal to alpha times theta minus sine theta.

OK, and now I was working out the relationship between θ and \tilde{t} . And let me just remind you that all I did was make sure that the upper limits of integration correspond to each other. So I'm just basically rewriting the upper limit of integration in terms of the new variable each time when we change variables starting from \tilde{t} going to \tilde{t} to x , and from x to θ . And then I use the equality of these two to write \tilde{t} in terms of θ . And these equations now hold with the subscript f present. When subscript f 's appear everywhere we can just drop them and say that the time that we called $t_{sub f}$ now we're just going to call t . And now we could put together our two final results which are maybe right over here. We have ct . I'll eliminate my tildes altogether now. ct , which is \tilde{t} , is equal to α times θ minus $\sin \theta$. And a divided by the square root of k , which is \tilde{t} and its sub f , but we're dropping the sub f , is equal to α times $1 - \cos \theta$. And this is as good as it gets.

Ideally, it would be nice if we could solve the top equation for θ as a function of t , and plug that into the bottom equation, and then we would get a as a function of t . That's what ideally we would love to have. But there's no way to do that analytically. In principle of course, this equation can be inverted. You could do it numerically. For any particular value of t you could figure out what value of θ makes this work, and then plug that value of θ into here. But there's no analytic way that you can write θ as a function of t . It's not a soluble equation.

So this is called a parametric solution in the sense that θ is a parameter. And as θ varies, both t and a vary in just the right way so that a is always related to t in the correct way to solve our original differential equation. That's what's meant by a parametric solution.

We can also see from this equation, or maybe from thinking back about how things are defined in the first place, how θ varies over the lifetime of our model universe. θ we discovered starts at 0. We discovered that when we wrote our integral over there. And we could also probably see it from here. We start our universe at $a = 0$. And at $a = 0$, we want $1 - \cos \theta$ to be 0 and $\theta = 0$ does that. So θ starts at 0, which corresponds to $a = 0$, and

it also corresponds putting $\theta = 0$ here to $t = 0$. So we have arranged things the way we intended so that $a = 0$ happens the same time $t = 0$ happens.

And then θ starts to grow. As θ grows, a increases, the universe gets bigger until θ reaches π . And when θ reaches π , $\cos \pi$ is -1 , you get a factor of 2 here, $2a$. That's as big as our universe gets. And then as θ continues beyond that, $1 - \cos \theta$ starts getting smaller again. So our universe reaches a maximum size when $\theta = \pi$, and then starts to contract. And then by the time $\theta = 2\pi$, you are back to where you started from, a is again equal to 0. We have universe which starts at 0 size, goes to a maximum size, goes back to 0 size, giving a Big Crunch at the end. And that's the way this closed universe behaves.

It turns out that these equations actually correspond to some simple geometry. It corresponds to a cycloid. And you may or may not remember what a cycloid is. There are the equations written on the screen, which are hopefully the same equations that I have on the blackboard. Can't always count on these things it turns out. But yeah, they are the same equations, that's healthy. And I have a diagram here which explains this cycloid correspondence. A cycloid is defined as what happens in this picture. Let me explain the picture. We have a disk shown in the upper left in its original position with a dot on the disk, which is initially at the bottom. And initially we're going to put this disk at the origin of our coordinate system to make things as simple as possible. And then we're going to imagine that this disk rolls without slipping to the right. And the path that this dot traces out, which is shown along that line, is a cycloid by definition. That's what defines a cycloid. It's the path that a point on a rolling disk evolved through.

So what I would like to do is convince you that this geometric picture corresponds to those equations. Let me put the equations higher to make sure everybody can see them. And it's actually not so complicated once you figure out how to parse the pictures. So what I've drawn in the upper right is a blow up of this disk after it's rolled through some angle θ . And I even made it the same angle θ in the two

cases. But this is just a bigger version of what's in the corner there showing the disk after it rolled a little bit. So after it's rolled through an angle θ , what we want to verify is that the horizontal and vertical coordinates here correspond to these two equations. And if they do, it means that we're tracing out the behavior of those two equations.

So look first at the horizontal component. The horizontal component is the x axis. So that should correspond to $\alpha \theta - \alpha \sin \theta$. And you can see that in the picture. We're talking about the horizontal component of the coordinates of this point P . And the point is that you can get to P starting at the origin. Now remember we're only looking at horizontal motion so we could start anywhere on that line. We could start at the origin, go to the right by $\alpha \theta$, and to the left by $\alpha \sin \theta$, and we get there. And that's exactly what this formula says. So if we just understand the $\alpha \theta$ and the $\alpha \sin \theta$ of those two lines we have it made.

So let's look first at what happens where this first arrow comes from, the $\alpha \theta$ line. That's just the total distance that the point of contact has moved during the rolling process. And the claim is that as something rolls, really the definition of rolling without slipping, is that the arc length that is swept out by the rolling is the same as the length along the surface on which it's rolling. You could imagine, if you like, that as it rolls there's a tape measure that's wrapped around it that gets left on the ground as it rolls. And if you could picture that happening, the existence of that movie really guarantees that the length on the ground is the same as what the length was when the tape measure was wrapped around the cylinder. So the length that the point of contact has moved is just $\alpha \theta$ the angle through which the disk is rolled. So that explains the $\alpha \theta$ label on that line.

To get the $\alpha \sin \theta$ on the line above, that's the distance between the point P and a vertical line that goes through the center of the disk. And that's just simple trigonometry on this triangle. The radius of our circle is α . And then by simple trigonometry, this length is $\alpha \sin \theta$, which is what the label says.

So to summarize what we've got here we can find the x-coordinate of the horizontal according to the point P by going to the right a distance $\alpha \theta$, and then back to the left the distance of $\alpha \cos \theta$. And that gives us an x-coordinate, which is exactly the formula that appears in the ct formula. So ct works. The horizontal component of that dot is just where it should be to trace out the equations that describe the evolution of a closed universe.

Similarly we can now look at the vertical components of that dot. Again it's most easily seen as the difference of two contributions. We're trying to reproduce this formula that says that's $\alpha \sin \theta$. So if we start at vertical coordinate 0, which means on the x-axis, we can begin by going up to the center of the disk. And the disk has radius α , so that's going up the distance α . And then we go down the distance of this piece of the triangle, going from the center of the disk to the point which is parallel to the point P. And that again is just trigonometry on that triangle, and it's $\alpha \cos \theta$ by simple trigonometry. So we can get to the elevation of point P by going up by α and down by $\alpha \cos \theta$. And that's exactly that formula. So it works. The x and y components, the horizontal and vertical components of that dot, are exactly the two formulas here. So one of them can be thought of as the x-axis, and one of them can be thought of as the y-axis. And the rolling of the disk just traces out the evolution of our closed universe. So closed universes evolve like a cycloid. Any questions about that?

OK, great. OK, Let me just mention that this angle θ is sometimes given a name. It's called the development angle of the universe or of the solution. And that is just intended to have the connotation that θ describes how developed the universe is and θ has a fixed scale. It always goes from 0 to 2π over the lifetime of this closed universe, no matter how big the closed universe might be.

That brings me to my next question I want to mention. How many parameters do we have in this solution? The way we've written it, it looks like it can depend on both α and k because both of them appear in the answer. And k is positive for our closed universes. These formulas will not make sense if k were negative, square

root of k appears there. We don't want anything to be imaginary. But k can have any value in principle and these equations would still be valid. So on the surface it appears like there's a two parameter class of closed universe solutions. But that's actually not true. Can somebody tell me why it's not true? Yes.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Exactly. Yes, since k has units of 1 over notch squared, you can change k to anything you want by changing your definition of a notch. And there's nothing fixed about the definition of a notch. So k is an irrelevant parameter. If we change k we're just rescaling the same solution, and not actually creating a new solution. So there's really only one parameter class of solutions. One could, for example, fix k to always be 1, and then we'd have a one parameter class of solutions indicated by α .

α , unlike k , really does have a clear, physical meaning related to the behavior of the universe. And we could see what that is if we ask, what is the total lifetime of this universe from beginning to end, from Big Bang to Big Crunch. We can answer that by just looking at the ct equation. From Big Bang to Big Crunch, we know that θ evolves from 0 to π back to 2π , which is the same as 0. So θ goes through one cycle of 2π during a lifetime of our model universe. As θ goes from 0 to 2π , $\sin \theta$ starts at 0 and eventually comes back to 0 when θ equals 2π . But θ increases from 0 to 2π over one cycle. So over one cycle of our universe, ct increases by α times 2π . So that tells us what the total lifetime of the universe is, I'll call it t_{total} . And we get it by just writing c times t_{total} equals 2π times α . And I think I can do this one without making a mistake. t_{total} is then $2\pi\alpha$ divided by c . And we can even check our units there. α has units of length, so length divided by c becomes a time, c being a velocity. So it has the right units. And that's the total lifetime of the universe. It's just determined by α . So α can be viewed as just the measure of the total lifetime of the universe, which can be anything for different sized closed universes.

α is also related to the maximum value of a over \sqrt{k} . And a has no fixed meaning, this is meters per notch. But a over \sqrt{k} does have units of meters. We

haven't yet really seen what it means physically, because that's related to the geometry of the closed universe which we'll be discussing later. But in any case as a mathematical fact, we could always say that the maximum value of a over \sqrt{k} is determined by α .

So a max over \sqrt{k} is equal to the maximum value of this expression. And this expression has a maximum value when θ equals π , which gives $\cos \theta$ equals -1 , which gives a^2 here. And that's as big as it ever gets, so that's just equal to 2α . So α is related in a very clear way to the total lifetime of our universe, and is also related to a over \sqrt{k} , although we haven't really given a physical meaning to a over \sqrt{k} yet. But we know it has dimensions of meters.

OK, the next calculation I want to do is to calculate the age of the universe as a function of measurable things. We learned for the flat matter dominated case that there was a simple answer to that. The age was just $2/3$ times the inverse Hubble parameter. So what you do now is get the analogous formula here, it follows in principle immediately from our description of the evolution. But we have to do a fair number of substitutions before we can really see how to express the age in terms of things that we're interested in.

The formula here tells us directly how to express the age of the universe. t is the age of the universe as a function of α and θ . But if you tell an astronomer to go out and measure α and θ so I could calculate the age, he says what in the world are α and θ . So what we'd like to do is to express the age in terms of things that astronomers know about. And the characterizations of the universe like this that an astronomer would know about would be the Hubble expansion rate, and some notion of the mass density. And the easiest way to talk about the mass density is in terms of ω , the fraction of the critical density that the actual mass density has.

So our goal is going to be to manipulate these equations. All the information is already there. But our goal will be to manipulate these equations to be able to express the age t in terms of h and ω .

OK, so first we need to remind ourselves what ω is. The critical density is defined as that density which makes the universe flat. And we've calculated that the critical density is equal to $3h^2 / 8\pi G$. We can then write the mass density ρ as ω times the critical density, which is just the definition of ω . ω is ρ divided by ρ_c , the actual mass density divided by the critical density. And putting what ρ_c is, we can express ρ as $3h^2 \omega / 8\pi G$. And being very pedantic, I'm just going to rewrite that in the form that we're actually going to use it by multiplying through. $8\pi G \rho$, taking these factors and bringing them to the other side, becomes just equal to $h^2 \omega$.

And you might recognize this particular combination as what appears in the Friedmann equation. The Friedmann equation told us that \dot{a}^2 / a^2 is equal to $8\pi G \rho / 3 - k / a^2$. And in order to get the substitutions that I want, I'm going to just rewrite this putting h^2 for \dot{a}^2 / a^2 . $8\pi G \rho$ we said we could write as $8h^2 \omega$. And then we have $-k / a^2$. Note that we really have here a \tilde{c}^2 . This is c^2 / k if I put them together. So this term can be written as $-c^2 / \tilde{c}^2$.

And this accomplishes one of our goals. It allows us to express \tilde{c} in terms of the quantities that we want in our answers, h and ω . And if we can do the same for θ , we have everything we need to express the age. So the implication here is that \tilde{c}^2 is equal to $c^2 / (h^2 \omega - 1)$.

To take the square root of that equation, to find out what \tilde{c} is, we need to think a little bit about signs and things like that. \tilde{c} is always positive. This is the scale factor divided by the square root of k , square root of k is positive, scale factors are always positive the way we defined it. So we can take the square root of that taking the positive square root of the right hand side. ω is bigger than 1 for our case, so $\omega - 1$ is a positive number, $8h^2$ is a positive number. So taking the square root there offers no real problem. We can write \tilde{c} is equal to c over,

I guess this point I might not notice until later. h can be positive or negative over the course of our calculation. We're going to talk about an expanding phase and a contracting phase. So when we take the square root of h squared, we want the positive number to give us a positive a . So it's the positive square root that we want, which is the magnitude of h , not necessarily h . When h is positive, the magnitude of h is h . h could be negative though, and the magnitude of h is still positive, and then the square root of $\omega - 1$, which is always positive. So that's our formula for a in terms of h and ω .

OK, now we want to evaluate α , and I guess I did not keep the formula for α quite as long as we needed it. When we defined α in the first place, let me remind you how it was defined, 4π over $3G\rho a^3$ over c^2 . And that can be evaluated using our formula for ρ and we'll put that in for ρ , and what we get is c over 2 times the magnitude of h using this formula for a as well. And then ω over $\omega - 1$ to the $3/2$ power. Just using this formula, and we know how to express ρ from the right hand side, and we know how to express a from this formula here. So everything's straightforward, and this is what we get.

And now I want to use these to rewrite this equation, a over the square root of k is equal to α times $1 - \cos\theta$. I'm going to replace a over root k by this formula. I'm going to replace α by that formula. So this implies, rewriting it, that c over the magnitude of h times the square root of $\omega - 1$ is equal to c over twice the magnitude of h times $\omega - 1$ to the $3/2$ power times $1 - \cos\theta$.

And now we've had to survive some boring algebra. But notice that now most things cancel away here. We get a very simple relationship between θ and h and ω , actually just ω . In particular, when we solve that, we get simply that $\cos\theta$ is equal to $2 - \omega$ over ω . So θ is directly linked to ω . If you know ω , you know θ , If you know θ , you know ω , by that formula. And we can rewrite this the other way around by solving for ω if we want. ω is equal to 2 over $1 + \cos\theta$.

Now we can look at this qualitatively to understand how ω is going to behave. At the very beginning $\cos \theta$ is equal to 1, θ is equal to 0, so ω is $\frac{2}{1+1}$, which is 1. So at very early times ω is driven to 1 even in a closed universe. As θ gets larger, $\cos \theta$ gets less than 1. This then becomes more than 1. So ω starts to grow as the universe starts to evolve. At the turning point, when the universe has reached its maximum size, θ is π , $\cos \theta$ is minus 1, ω is infinite at the turning point. That may or may not be a surprise. But if you think about it, it's obvious. At the turning point h is 0, therefore the critical density is 0, but the actual density is not 0. And the only way the actual density can be 0 while the critical density is 0 is for ω to be infinite, so we should have expected that. And then the return trip, the collapsing phases, a mirror image of the expanding phase, ω goes from infinity at the turning point back to 1 at the moment of the Big Crunch. Yes.

AUDIENCE: I'm confused with what the universe would look like when it gets to infinity.

PROFESSOR: It would look static. It's temporarily static. It's reached a maximum size and is about to turn around and collapse, so h is 0.

AUDIENCE: OK, but like with the density.

PROFESSOR: Well we could calculate the density. It's some number which depends on α .

AUDIENCE: OK, but that doesn't diverge or anything.

PROFESSOR: It doesn't diverge or anything, no. It's just some finite density. At the turning point, it's just a finite density that can be expressed in terms of α . Sounds like a good homework problem. I think maybe I'll do that.

OK, so this now allows us to express θ as a function of ω , which is what we wanted. If we express θ as a function of ω , and α as a function of ω and h , we have our answer. We have t expressed as a function of h and θ .

So there are choices about how exactly to express ω in terms of θ that will

involve inverse trigonometric functions. And anything that can be expressed in terms of an inverse cosine, can also be expressed as an inverse sine by doing a little bit of manipulations. We have our choice here about what we want to do. But in any case, the answer already has a factor of sine theta in it. So it's most useful to express theta as the inverse sine of what we get from that formula, to express the answer in what at least to me is the simplest form. So I'm going to manipulate this a little bit to find out what sine theta is in terms of omega, and then invert that to express theta in terms of omega.

So sine theta is of course plus or minus the square root of 1 minus cosine squared theta. And cosine theta we know in terms of omega, so we can express this in terms of omega. And if you do that it's straightforward enough algebra. It's plus or minus, depending on which sign of the square root is relevant, and we'll talk about that in a minute. It's plus or minus the square root of 2 times omega minus 1 over omega. So we can express theta as the inverse sine of this expression.

And now what I want to do is to make use of this to put into this expression using the value for alpha that we've already calculated and wrote over there. So we get our final answer, which is that t is equal to omega over twice the magnitude of the Hubble expansion rate times omega minus 1 to the 3/2 power times the arc sine or inverse sine of twice the square root of omega minus 1 over omega. That's just the theta that appears in this formula written in terms of sine theta or the inverse sine of the quantity that we determined was the sine of theta. And then I see here I should have a plus or minus because we haven't figured out our signs yet. Either is actually possible, depending on where we are in the evolution. And then minus or plus twice the square root of omega minus 1 over omega. That's a plus or minus, this is a minus or plus. And the reason I wrote one upstairs and one downstairs is that we don't yet really know how to evaluate theta, but the sign of this term is always going to be the negative of the sign of that term, this or that minus sign. So theta can be the inverse sine of this expression with either sign of the sine. Sorry for the puns. But whichever it is, it's the same on there as it is here but with a minus sign in front, that minus sign.

OK, so this is our final formula for the age. But we still need to think a little bit about the s-i-g-n signs of these inverse functions that appear here, that and that, and straightforward if you just take it case by case. And in the notes we have a table which I'll put shortly on the screen. But let's start by just talking about the earliest phase where the universe is shortly after the Big Bang, so that the development angle is a small angle.

We know that theta is going to go from 0 to 2 pi over the lifetime of our universe. So I now want to think of theta being small. And small means small compared to any number you think about. So when theta is small the sine theta is nearly theta. Both are positive. And in that case the sine theta being positive means this is the positive root in this equation, and therefore the positive root in that equation. So for early times this would be the plus sign, and that would mean that this would be the minus sign. Again the minus sign just coming from there and theta being positive. So for early times it's a plus and minus. And the arc sine is itself ambiguous. For early times the angle we know is going to be just a little bit bigger than 0. So that's the evaluation that we make of the arc sine function. Pi plus that would also give us the same sine. It would be another possible value for the arc sine. And of course 2 pi plus that would also be another possible root. So you have to know which root to take to know the right answer here because as an angle, 0 is equivalent to 2 pi, but as a time, 0 is not at all equivalent to 2 pi. So you do have to know the right one to take. We'll continue doing that on a case by case basis.

Those are the equations, that's the formula for the age, and that's the formula for the age with a description of which roots to take for each case, which just comes out by following the evolution, we know that theta is going from 0 to 2 pi, and this last column, the inverse sine of the expression which means the expression that appears here. For the smallest angle is 0 to pi over 2. We can think of this actually as defining our columns. Theta starts at 0 so the time lengths between 0 and pi over 2, a time length between pi over 2, a time length between pi and 3 pi over 2, and a final time length between 3 pi over 2 and 2 pi. And the first two correspond to the expanding phase, second two correspond to the contracting phase. We can easily see what values of omega are relevant in those cases.

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Omega we said starts at 1 and gets larger, the borderline where the angle is π over 2 one could just plug into these formulas and see amounts to omega equals 2. So that is a division line between these first two quadrants just calculated from the value of theta. Omega then goes to infinity as we said, comes backwards and back to 1 in the end. And in this column we just figure out which sign choice corresponds to getting the right value for omega and the angle that appears in the arc sine of our formula for the age, the formula there.

So any one of these I claim is very obvious. Seeing the whole picture takes time because I think you really have to look at each case one at a time to make sure you understand it in detail. But if you understand the initial expanding phase that's what corresponds to our universe if our universe were closed. And the others are just as easy. You just have to take them one at a time I think.

OK, we're going to end there. We will continue on Thursday.