8.321 Quantum Theory-I Fall 2016

Midterm quiz

19 Oct 2016

Useful facts.

$$\int du e^{-au^2+ibu} = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}} \tag{1}$$

Ground state wave function of a 3d simple harmonic oscillator of mass m, frequency ω is

$$\begin{aligned} \psi_0(\vec{r}) &= N_0 e^{-\frac{r^2}{2x_0^2}} \\ x_0 &= \sqrt{\frac{\hbar}{m\omega}} \\ N_0 &= \frac{1}{\pi^{\frac{3}{4}} x_0^{\frac{3}{2}}} \end{aligned}$$

1. **(20 points)**

   Consider a two level system with states $|0\rangle$ and $|1\rangle$. The Hamiltonian is known to be

   $$H = a|0\rangle\langle 1| + b|1\rangle\langle 0| \tag{2}$$

   (a) Write the condition on a, b for this to be a legal Hamiltonian. Assume this condition is satisfied for the rest of the problem.

   (b) What are the eigenvalues of H?

   (c) The system is prepared in the state $|0\rangle$ at $t = 0$. What is the probability that it will be found in state $|1\rangle$ after a time t?

2. **(20 points)**

   A spin-1/2 particle moves in three dimensions. A convenient basis for the Hilbert space are the states $|\vec{p}, S_z = +\frac{\hbar}{2}\rangle, |\vec{p}, S_z = -\frac{\hbar}{2}\rangle$ where $\vec{p}$ is the momentum, and S^z is the z-component of the spin. Suppose that the Hamiltonian is known to be

   $$H = \frac{p^2}{2m} - v\vec{\sigma}\cdot\vec{p} \tag{3}$$

   Here $\vec{\sigma}$ are the usual Pauli matrices that act on the spin degrees of freedom. (m is a mass and v a positive constant with dimensions of velocity.)

   (a) Is H diagonal in the basis described above?

   (b) Find the energy eigenvalues.

   (c) Find the corresponding energy eigenfunctions.

   (d) Find the ground state energy. Is the ground state unique?

3. **(30 points)**

   Consider a system with a total of k possible states with a basis labellled $|n\rangle$. $n = 1, \ldots\ldots, k$. In the expressions below we define $|k+1\rangle \equiv |1\rangle$. Consider two operators R_1 and R_2 defined in this space with the following properties:

   $$R_1|n\rangle = |n+1\rangle \tag{4}$$
   $$R_2|n\rangle = e^{\frac{2\pi i n}{k}}|n\rangle \tag{5}$$

   (a) Is either of R_1 or R_2 hermitian? Are either of them unitary?

   (b) R_2 is diagonal in the $|n\rangle$ basis, and has eigenvalues $e^{\frac{2\pi i n}{k}}$. Find the eigenvalues of R_1.

   (c) Show that $R_{1,2}$ satisfy

   $$R_1R_2 = e^{-\frac{2\pi i}{k}}R_2R_1 \tag{6}$$

   (d) Suppose there is a Hermitian operator A in this system that commutes with both R_1 and R_2. What can you say about A?

4. **(30 points)**

A particle of mass m is initially in the ground state of a three dimensional harmonic trap described by the potential $V(\vec{r}) = \frac{1}{2}m\omega^2 r^2$. At time $t = 0$, the trap is suddenly released. The subsequent motion is determined by the free Hamiltonian. The questions below all refer to times $t > 0$.

(a) What are $\langle \vec{r} \rangle, \langle r^2 \rangle, \langle \vec{p} \rangle, \langle p^2 \rangle$ as functions of t?

(b) Write down the momentum space wavefunction $\phi(\vec{p}, t)$? What is the probability distribution $\mathcal{P}(\vec{p}, t)$ of the momentum $\vec{p}$?

(c) What is the position space wavefunction $\psi(\vec{r}, t)$? What is the probability $P(\vec{r}, t)$ of the position $\vec{r}$?

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