8.321 Quantum Theory-I Fall 2015

Midterm quiz

28 Oct 2015

Useful facts.

$$\int du e^{-au^2} = \sqrt{\frac{\pi}{a}} \tag{1}$$

Ground state wave function of a 1d simple harmonic oscillator of mass m, frequency ω is

$$\begin{aligned} \psi_0(x) &= N_0 e^{-\frac{x^2}{2x_0^2}} \\ x_0 &= \sqrt{\frac{\hbar}{m\omega}} \\ N_0 &= \frac{1}{\pi^{\frac{1}{4}}\sqrt{x_0}} \end{aligned}$$

1. **(20 points)**

   (a) For a hermitian operator G, suppose that $\langle a|G|a\rangle = 0$ for all $|a\rangle$. Show that $G = 0$.

   (b) Prove part (a) for an arbitrary operator G by writing it as the sum of a Hermitian operator and an anti-Hermitian operator.

   (c) Prove that, given operators A and B, if $\langle a|A|a\rangle = \langle a|B|a\rangle$, then $A = B$.

   (d) Prove that, if $\langle a|G|a\rangle = 1$ for all normalized $|a\rangle$, then $G = 1$.

2. **(20 points)**

An atom can be in one of two internal states - the ground state $|g\rangle$ and an excited state $|e\rangle$. The energy difference between these two states is Δ. The atom is prepared at $t = 0$ in a state

$$|\theta_0; 0\rangle = \frac{1}{\sqrt{2}} \left(|g\rangle + e^{i\theta_0} |e\rangle \right) \tag{2}$$

(a) Calculate the mean value of the energy in this state relative to the ground state.

(b) Under time evolution show that the state retains the same form with a time dependent θ, *i.e*, $\theta = \theta(t)$ (upto a time dependent phase). Calculate $\theta(t)$ explicitly.

(c) Show by explicit calculation with the state $|\theta(t); t\rangle$ that the mean energy does not change with t.

3. **(30 points)**

Consider a particle moving in a circle of radius R. If we denote the position of the particle as x we must regard x and $x + 2\pi R$ as the same physical location. Wavefunctions in the Hilbert space of this particle must be periodic:

$$\psi(x + 2\pi R) = \psi(x) \tag{3}$$

Consider plane wave states $|p\rangle$ with definite momentum p defined through their wave function

$$\langle x|p\rangle = A e^{\frac{ipx}{\hbar}} \tag{4}$$

(a) Show that the periodicity restriction on the Hilbert space forces p to take certain discrete values p_n. Find the allowed p_n and the corresponding normalized wave functions.

(b) The free particle Hamiltonian is

$$H = \frac{p^2}{2m} \tag{5}$$

Obtain the energy eigenvalues and their degeneracies.

(c) Define the operator O through

$$O|x\rangle = R e^{-\frac{ix}{R}} |x\rangle \tag{6}$$

What is $O|p_n\rangle$? What is $O^\dagger|p_n\rangle$?

(d) Compute the commutator $[O, p]$.

4. **(30 points)**

Consider a quantum particle in a one dimensional oscillator potential with the Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \tag{7}$$

The particle is initially in its ground state. Starting time $t = 0$ the particle suddenly is subjected to a constant force F_0 leading to an extra term in the Hamiltonian

$$H_1 = -F(t)x \tag{8}$$

(The 'force' $F(t)$ is 0 for $t < 0$, and is F_0 for $t > 0$.)

(a) What is the ground state wave function in the position representation of the new Hamiltonian that describes the system for $t > 0$?

(b) Write the same ground state wave function in momentum representation.

(c) Assume that at $t = 0^+$ the particle is in the same state it was in for $t < 0$. It's subsequent time evolution is determined by the new Hamiltonian. Calculate the probability that if the state of the particle is measured at $t = 0^+$, it will be found in the ground state of the new Hamiltonian?

(d) Calculate $\langle x \rangle$, $\langle p \rangle$ as a function of time for $t > 0$.