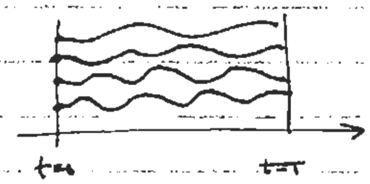
6.13 Adiabatic Theorem & Berry's phase (Sakurai: 464-480)

Time-dependent $H(t)$ with

$$H(t)\,|n,t\rangle = E_n(t)\,|n,t\rangle$$

Assume levels never cross



Adiabatic theorem:

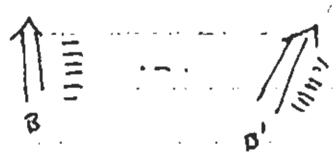
Start in state $|i\rangle$, $\quad H(0)\,|i\rangle = E_i(0)\,|i\rangle$

If $H(t)$ varies slowly, $\quad |\psi,t\rangle = e^{i\alpha(t)}\,|i,t\rangle$

(state stays at same level, only phase changes)

Basically, H must changes slowly compared to natural oscillation rates in problem
$\dot{H}/H \ll \omega$

Example: spin S particle in changing $\vec{B}$ field



Quantitative understanding:

Expand $\quad |\psi,t\rangle = \sum C_n(t)\,|n,t\rangle \qquad \langle n,t|m,t\rangle = \delta_{nm}$

$$i\hbar \frac{d}{dt}\left(\sum C_n(t)\,|n,t\rangle\right) = \sum_n E_n(t)\,C_n(t)\,|n,t\rangle$$

$$\Rightarrow \quad i\hbar\,\dot{C}_m(t) + i\hbar \sum_n C_n(t)\,\langle m,t|\frac{d}{dt}|n,t\rangle = C_m(t)\,E_m(t)$$

$$i\hbar \dot{C}_m(t) = C_m(t) E_m(t) - i\hbar C_m(t) \langle m,t| \frac{d}{dt} |m,t\rangle \underbrace{- \sum_{n \neq m} i\hbar C_n(t) \langle m,t| \frac{d}{dt} |n,t\rangle}_{\text{assume small} \quad \{\text{justify presently}\}}$$

$$\dot{C}_m(t) = \left(-\frac{i}{\hbar} E_m(t) - \langle m,t| \frac{d}{dt} |m,t\rangle \right) C_m(t)$$

pure imaginary, since $\frac{d}{dt} \langle m|m\rangle = \langle \dot{m}|m\rangle + \langle m|\dot{m}\rangle = 0$

$$C_m(t) = C_m(0)\, e^{\underbrace{-\frac{i}{\hbar}\int_0^t dt' E_m(t')}_{\text{dynamical phase}} + \underbrace{\int_0^t dt' \langle m,t'| \frac{d}{dt'} |m,t'\rangle}_{\text{Berry's phase (geometrical)}}}$$

Note: phase of basis $|m,t\rangle$ can be chosen arbitrarily, changes Berry's phase.

Can set $|m,t\rangle = e^{\beta_m(t)} |\widetilde{m,t}\rangle$ so that $\langle \widetilde{m,t}| \frac{d}{dt} |\widetilde{m,t}\rangle = 0$.

Why is $C_n(t) \langle m,t| \frac{d}{dt} |n,t\rangle$ small, $m \neq n$?

Fix $\langle m,t| \frac{d}{dt} |m,t\rangle = 0$

Take $C_m(t) = \tilde{C}_m(t)\, e^{-\frac{i}{\hbar}\int_0^t dt' E_m(t')}$

$$i\hbar \dot{\tilde{C}}_m(t) = -i\hbar \sum_{n \neq m} \tilde{C}_n(t) \langle m,t| \frac{d}{dt} |n,t\rangle\, e^{-\frac{i}{\hbar}\int_0^t (E_n(t') - E_m(t'))\, dt'}$$

But $$\frac{d}{dt}\left(H(t)|n,t\rangle = E_n(t)|n,t\rangle\right)$$

$$\Rightarrow \frac{dH(t)}{dt}|n,t\rangle + H(t)\frac{d}{dt}|n,t\rangle = \frac{dE_n(t)}{dt}|n,t\rangle + E_n(t)\frac{d}{dt}|n,t\rangle$$

$$\Rightarrow \langle m,t|\frac{dH}{dt}|n,t\rangle = \left(E_n(t) - E_m(t)\right)\langle m,t|\frac{d}{dt}|n,t\rangle$$

$$\Rightarrow \langle m,t|\frac{d}{dt}|n,t\rangle = \frac{\langle m,t|\frac{dH}{dt}|n,t\rangle}{E_n(t) - E_m(t)}$$

Assume $\tilde{C}_n$, $|n,t\rangle$, $\frac{dH}{dt}$, $E_n(t)$ slowly varying, treat as constant

$$\dot{\tilde{C}}_m(t) = \sum_{n\neq m}\frac{\langle m|\frac{dH}{dt}|n\rangle}{\hbar\omega_{mn}}e^{i\omega_{mn}t}\tilde{C}_n$$

$$\tilde{C}_m(t) = \sum_{n\neq m}\frac{\langle m|\frac{dH}{dt}|n\rangle}{i\hbar\omega_{mn}^2}\left(e^{i\omega_{mn}t} - 1\right)\tilde{C}_n$$

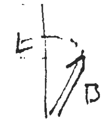
Amplitude oscillates.

If $\boxed{\hbar\langle m|\frac{dH}{dt}|n\rangle \ll (E_m - E_n)^2}$, small effect.

this is regime where adiabatic approximation is valid.

<u>Example</u>: Spin 1/2 particle in rotating B field

$$\vec{B}(t) = B(\sin\theta\cos\phi(t), \sin\theta\sin\phi(t), \cos\theta)$$



$$H(t) = 2\vec{B}\cdot\frac{\vec{S}}{\hbar} = \vec{B}\cdot\vec{\sigma} = B\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}$$

Eigenstates

$$|+,t\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \; e^{i\phi(t)} \end{pmatrix} \qquad E_+ = B$$

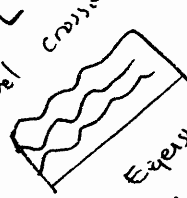
$$|-,t\rangle = \begin{pmatrix} \sin\theta/2 \\ -\cos\theta/2 \; e^{i\phi(t)} \end{pmatrix} \qquad E_- = -B$$

$$\frac{d}{dt} H(t) = B \begin{pmatrix} 0 & -i \\ i\dot\phi \sin\theta \, e^{i\phi} & \end{pmatrix}$$

When is adiabatic ap

$\langle +,t| \frac{dH}{dt}$

Adiabatic thm: if $\dot H / H \ll \omega$ & no level crossing $\left[\hbar \langle m | \frac{dH}{dt} | n \rangle \ll (E_m - E_n)^2 \right]$



Eigenstate of $H(t)$, $[H(t)|n,t\rangle = E_n(t)|n,t\rangle]$

$|\psi,0\rangle = |i\rangle$

$|\psi,t\rangle = e^{i\alpha(t)} |i,t\rangle$

$\alpha(t) = \underbrace{-\frac{1}{\hbar}\int_0^t dt' \, E_i(t')}_{\text{dyn. phase}} + \underbrace{\int_0^t dt' \langle m,t'| \frac{d}{dt} |m,t'\rangle}_{\text{geometrical (Berry's) phase}}$

$\begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \; e^{i\phi} \end{pmatrix}$ (2)

$E_+ - E_- =$

so adiabatic approx good w

$$\hbar \left| \langle m | \frac{dH}{dt} | n \rangle \right| \ll \min |E_m - E_n|^2$$

$$\Leftrightarrow \quad \hbar B |\dot\phi \sin\theta| \ll 4B^2$$

$$\Leftrightarrow \quad \hbar |\dot\phi \sin\theta| \ll 4B$$

Take adiabatic approx.. assume initial state $|i,0\rangle = |+,0\rangle$

$$i\hbar \dot{C}_+ = \cancel{4/\hbar}\left(B - i\hbar \underbrace{\langle +,t | \tfrac{d}{dt} | +,t\rangle}_{\sin^2\theta/2 \; i\dot{\phi}} \right) C_+$$

$$\dot{C}_+ = \left(-\frac{i}{\hbar} B - i\dot{\phi}\sin^2\theta/2 \right) C_+$$

$$C_+(t) = e^{-\frac{i}{\hbar}Bt \; - \; i\phi\sin^2\theta/2} \; |+,t\rangle$$

dynamical phase

Berry's phase
[note: independent of how ϕ changed over time; depends only on $\phi(t)$.]

For constant rate (exactly solved case)

$$\phi = \frac{2\pi t}{T} = \omega t \qquad \omega = \frac{2\pi}{T}$$

Adiabatic approx good when $\hbar\dot{\phi}\sin\theta \ll 4B$

$$\Leftrightarrow \frac{\hbar}{T} \ll B \;, \qquad T \gg \frac{\hbar}{B}.$$

Berry's phase

Consider H depending on parameter $R(t)$,

R in some space X

case of particular interest: $\vec{R} \in \mathbb{R}^3$
(e.g. $\vec{R}$ is B-field)

Basis $|n(R)\rangle$:

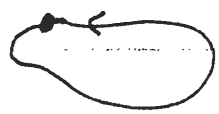
$$H(R)\,|n(R)\rangle = E_n(R)\,|n(R)\rangle .$$

Vary R slowly, so adiabatic approx. is valid.
If $|\psi,0\rangle = |n(R(0))\rangle$,

$$|\psi,t\rangle = e^{-\frac{i}{\hbar}\int_0^t E(t')dt' + i\gamma_n(t)}\;|n(R(t))\rangle .$$

where
$$\dot{\gamma}(t) = i\,\langle n(R(t))|\frac{d}{dt}|n(R(t))\rangle$$
$$= i\langle n(R(t))|\frac{\partial}{\partial R^j}|n(R(t))\rangle\,\frac{\partial R^j}{\partial t}$$

Consider taking R around a closed loop in X



$R(t)$

$$R(T) = R(0)$$

Stokes:
$$\oint_{\partial S} \omega = \int_S d\omega$$

∂S: p-dimensional boundary of a (p+1)-volume S

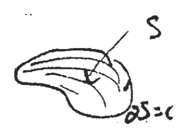
ω: p-form

S: (p+1)-volume

$d\omega$: (p+1) form

for $p=1$:
$$\oint_{C=\partial S} \omega_i\, dR^i = \iint_S \partial_{[i}\omega_{j]}\, d\sigma^i d\sigma^j$$

antisymmetrize on i,j



If $X = \mathbb{R}^3$, describe with vector calculus

$$\oint_C \vec{\omega} \cdot d\vec{R} = \iint (\vec{\nabla} \times \vec{\omega}) \cdot d\vec{S}$$

So

$$\gamma_n = i \iint d\sigma^i d\sigma^j \, \partial_i \langle n(R) | \partial_j | n(R) \rangle$$

$$\cancel{= - \iint \vec{V}_n(R) \cdot d\vec{S} \quad \text{if } X = \mathbb{R}^3}$$

Note that phase only depends on curve C, not on $R(t)$.

If $X = \mathbb{R}^3$,

$$\gamma_n = - \iint \vec{V}_n(R) \cdot d\vec{S}$$

$$\begin{aligned} V_n^i(R) &= \varepsilon^{ijk} \, \mathrm{Im} \, \partial_j \langle n(R) | \partial_k | n(R) \rangle \\ &= \varepsilon^{ijk} \, \mathrm{Im} \left(\partial_j \langle n(R) | \right) \left(\partial_k | n(R) \rangle \right) \\ &= \sum_m \varepsilon^{ijk} \, \mathrm{Im} \left(\partial_j \langle n(R) | m \rangle \langle m | \partial_k | n(R) \rangle \right) \end{aligned}$$

But

$$\langle m | \frac{\partial}{\partial R^i} | n(R) \rangle = \frac{\langle m | \frac{\partial H}{\partial R^i} | n(R) \rangle}{E_n(R) - E_m(R)}, \quad m \neq n$$

(as with $\frac{d}{dt}$)

So

$$\boxed{V_n(R) = \varepsilon^{ijk}\, \mathrm{Im} \sum_{m\neq n} \frac{\langle n(R)| \frac{\partial H}{\partial R^i} |m\rangle \langle m| \frac{\partial H}{\partial R^j} |n(R)\rangle}{(E_n - E_m)^2}} \quad (*)$$

Gives Berry's phase through

$$\boxed{\delta_n(C) = -\iint_S \vec{V}_n(R) \cdot d\vec{S}}$$

For more general X,

$$\delta_n = -\iint d\sigma^i\, d\sigma^j\, \Omega_{ij}$$

$$\Omega_{ij} = \mathrm{Im} \sum_{m\neq n} \frac{\langle n(R)| \frac{\partial H}{\partial R^i} |m\rangle \langle m| \frac{\partial H}{\partial R^j} |n(R)\rangle}{(\epsilon_n(R) - \epsilon_m(R))^2}$$

Notes: [note: more restrictive than $\beta_n(t)$!.]

*) Redefining phases $|n(R)\rangle \to e^{i\beta_n(R)} |n(R)\rangle$ doesn't change $V_n(R)$ or $\delta_n(C)$.

*) $\vec{\nabla} \cdot \vec{V}_n(R) = 0$ [$ddw = 0$, show explicitly for (*) in HW]

*) if path encloses no area $\delta_n(C) = 0$ 

*) cannot pass through degeneracy pt $\epsilon_n(R) = \epsilon_m(R)$.