6.2 Exactly solvable 2-state problem

Consider a two-state system with

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$V(t) = \begin{pmatrix} 0 & \gamma e^{i\omega t} \\ \gamma e^{-i\omega t} & 0 \end{pmatrix} \quad [\, V_{12} = \gamma e^{i\omega t};\ V_{21} = \gamma e^{-i\omega t} \,]$$

In interaction picture

$$i\hbar \dot{C}_1 = \gamma e^{i[\omega + \frac{E_1 - E_2}{\hbar}]t} \, C_2(t)$$

$$i\hbar \dot{C}_2 = \gamma e^{i[-\omega - \frac{E_1 - E_2}{\hbar}]t} \, C_1(t)$$

$$\Rightarrow \qquad \frac{dc}{dt} = \frac{-i\gamma}{\hbar} \begin{pmatrix} 0 & e^{i(\omega - \omega_{21})t} \\ e^{-i(\omega-\omega_{21})t} & 0 \end{pmatrix} C(t) \qquad (*)$$

where $C(t) = \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix}$, $\quad \omega_{21} = \frac{E_2 - E_1}{\hbar}$

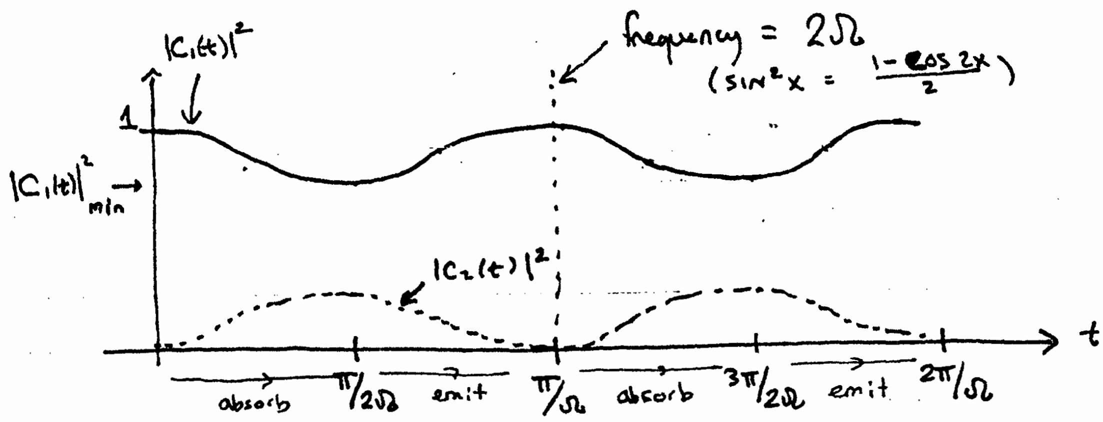
Can find exact solution of $(*)$. [HW]

With initial conditions $C_1(0) = 1$, $C_2(0) = 0$,

$$|C_2(t)|^2 = \frac{\gamma^2}{\gamma^2 + \hbar^2(\omega - \omega_{21})^2/4} \sin^2 \Omega t$$

$$\Omega = \sqrt{\frac{\gamma^2}{\hbar^2} + \frac{(\omega - \omega_{21})^2}{4}}$$

$$|C_1(t)|^2 = 1 - |C_2(t)|^2$$

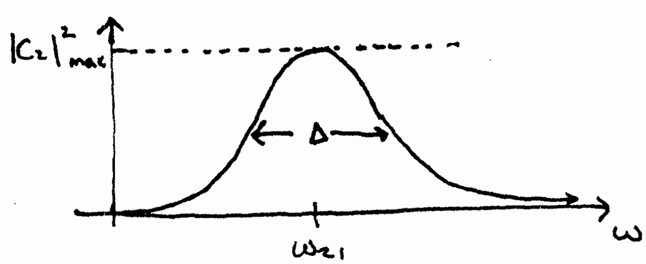


$$|C_1(t)|^2_{min} = \frac{(\omega - \omega_{21})^2}{(\omega-\omega_{21})^2 + 4\delta^2/\hbar^2}$$

At resonance, $\omega = \omega_{21}$

$\omega\Omega = \delta/\hbar$, $\qquad |C_1(t)|^2_{min} = 0.$

Amplitude as function of ω:



Δ = full width @ half max
$= 4\delta/\hbar$

- Amplitude peaked @ resonance
- width $\propto \delta$ (strength of perturbation)

Time-dep. interaction potentials

$$H = H_0 + V(t)$$

I. picture

$$|\psi(t)\rangle_S = e^{iH_0 t/\hbar} \; |\psi(t)\rangle_I = \sum c_n(t) |n\rangle$$

eigenstates of H_0

EOM

$$i\hbar \dot{c}_n(t) = \sum V_{nm}(t)\, e^{i\omega_{nm} t}\, c_m(t)$$

$$V_{nm}(t) = \langle n | V_S(t) | m \rangle$$

$$\omega_{nm} = \frac{E_n - E_m}{\hbar} = -\omega_{mn}$$

2-state system:

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$V(t) = \begin{pmatrix} 0 & \gamma e^{i\omega t} \\ \gamma e^{-i\omega t} & 0 \end{pmatrix}$$

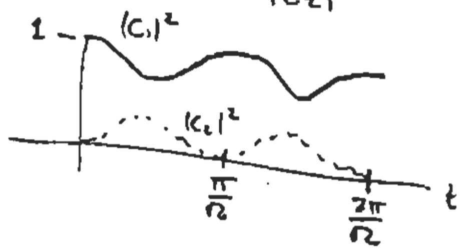
IC's $c_1(0) = 1$, $c_2(0) = 0$

$$|c_2(t)|^2 = \frac{\gamma^2}{\gamma^2 + \hbar^2(\omega - \omega_{21})^2/4} \sin^2 \Omega t$$

$$\omega_{21} = \frac{E_2 - E_1}{\hbar}$$

$$\Omega = \sqrt{\gamma^2/\hbar^2 + \frac{(\omega - \omega_{21})^2}{4}}$$

$$|c_1|^2 = 1 - |c_2|^2$$



- Periodically forced 2-state system is a basic problem
  - demonstrates fundamental features of absorbtion & emission.

Analogous to absorbtion & emission of radiation by particles in EM fields

- simplify atom to 2-level system $\frac{\overline{\quad} E_2}{\overline{\quad} E_1}$
- Couple to background rad. field @ frequency ω

$$\left(V \sim \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix} \right)$$

- when ω near $\omega_{21} = \frac{E_2 - E_1}{\hbar}$, system can absorb a quantum of radiation from BG field
- same with stimulated emission when in E_2.

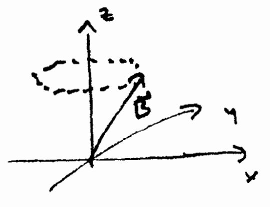
Again, we are doing semiclassical approximation, complete picture of spont. emission requires quantizing b.g. field.

Examples of 2-state systems

a) Spin magnetic resonance

Consider spin $1/2$ particle $(|+\rangle, |-\rangle)$ in magnetic field

$$B = B_0 \hat{z} + B_1 (\hat{x} \cos \omega t + \hat{y} \sin \omega t)$$



$$H = -g \mu_B \frac{\vec{S}}{\hbar} \cdot \vec{B}$$
$$(2)\left(\frac{e\hbar}{2mc}\right)\left(\frac{\vec{\sigma}}{2}\right)$$

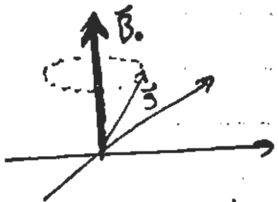
$$= H_0 + V(t)$$

$$H_0 = -\frac{eB_0\hbar}{2mc}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$V(t) = -\frac{eB_1\hbar}{2mc}\begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix}$$

Can now apply previous general discussion.

<u>Without</u> B_1,



spin precesses at $\omega = \frac{eB_0}{mc}$ [from last semester]

[phases of C_+, C_- move in opp. direction]

$|C_+|$, $|C_-|$ unchanged – only effect in phases.

$\langle S_z \rangle$ unchanged.

<u>Including</u> B_1, as above, gives oscillations betw. $|C_+|^2, |C_-|^2$.
(spin – flops)

[Classically – precession about a t-dependent axis]

At resonance, $\vec{B}$ rotates @ $\omega = \omega_{21} = \frac{eB_1}{mc}$ [3.98]
– same rate as precession about B_0.
$\Rightarrow$ spin goes all the way down.

Notes:

i) transitions $|+\rangle \to |-\rangle$ occur for <u>any</u> B_1, even very small.

ii) In practice, easier to make

$$\vec{B} = (0,\ B_1 \cos\omega t,\ B_0)$$

$$\Rightarrow e^{i\omega t} \text{ term } + e^{-i\omega t} \text{ term.}$$

Near resonance $\omega = \omega_{21}$ relevant, other is irrelevant, so same <u>physical</u> effects.

b) MASERS

Ammonia NH_3 molecule: 2 nearby states $|A\rangle$, $|S\rangle$ > small ΔE

Under parity operator P: $x \to -x$,

$$P|A\rangle = -|A\rangle$$
$$P|S\rangle = |S\rangle$$

Electric dipole moment $\vec{\mu}_{el}$ odd under parity: $P\vec{\mu}_{el}P = -\vec{\mu}_{el}$.

Thus $\langle S|\mu_{el}|S\rangle = \langle A|\mu_{el}|A\rangle = 0$
while $\langle S|\mu_{el}|A\rangle = \langle A|\mu_{el}|S\rangle \neq 0$.

Interaction with E field: $V = -\vec{\mu}_{el} \cdot \vec{E}$

Consider $\vec{E} = |E|_{max}\, \hat{z} \cos\omega t$

Gives example of 2-state problem.

MASER: select beam of $|A\rangle$'s

pass through μwave field $\omega = \frac{E_A - E_S}{\hbar}$ for time $t = \frac{\pi\hbar}{2\epsilon}$.

All $|A\rangle \to |S\rangle$, amplifies field

MASER = Microwave Amplification by Stimulated Emission of Radiation

Similar story w/ Hydrogen maser using hyperfine 21 cm line [this is where 1420 MHz known]