6.11 Planck's radiation law

Consider an atom in a radiation field which goes between states A & B by emission/absorbtion

$$A \rightleftarrows B + \gamma$$

In thermal equilibrium.

$$N(A)\cdot W_{emis} = N(B)\cdot W_{abs}$$

$$\frac{N(B)}{N(A)} = \frac{e^{-E_B/kT}}{e^{-E_A/kT}} = e^{+\hbar\omega/kT} = W_{emis}/W_{abs} = \frac{\Pi_{k,\alpha}+1}{\Pi_{k,\alpha}}$$

so

$$\Pi_{k,\alpha}\left(e^{+\hbar\omega/kT}-1\right) = 1$$

$$\Pi_{k,\alpha} = \frac{1}{e^{+\hbar\omega/kT}-1}$$

Energy density per unit volume

$$U(\omega)\,d\omega = \frac{1}{L^3}\cdot \underbrace{2}_{\text{polarization}} \cdot \frac{\hbar\omega}{e^{\hbar\omega/kT}-1}\cdot \rho(\omega)$$

$$\rho(\omega) = \hbar\rho(E) = 4\pi\left(\frac{L}{2\pi c}\right)^3\omega^2$$

$$= \frac{8\pi\hbar}{c^3}\left(\frac{\omega}{2\pi}\right)^3\left(\frac{1}{e^{\hbar\omega/kT}-1}\right)d\omega$$

in terms of $\nu = \omega/2\pi$

$$U(\nu)\,d\nu = \frac{8\pi h}{c^3}\nu^3\frac{1}{e^{h\nu/kT}-1}\,d\nu$$

Planck law (Planck: 1900)

6.12 Damping & natural line width

Back to TDPT

$$H = H_0 + V\cancel{(t)} \qquad \text{(assume } V \text{ t-independent)}$$

$$|\psi(t)\rangle_I = \sum C_n(t)\,|n\rangle$$

$$i\hbar\,\dot{C}_n = \sum_m V_{nm}\, e^{i\omega_{nm}t}\, C_m(t)$$

1st order approx: replace $C_m(t) \to \delta_{m0}$ on RHS

Better approximation (for unstable states) (Weisskopf & Wigner):

Assume $\quad C_i(t) = e^{-\gamma/2\, t}$

$$\gamma = \gamma_1 + i\gamma_2$$

γ_2 = energy shift (pure phase $e^{-i\gamma_2/2\,t}$)

γ_1 : describes decay rate ($|C_i|^2 = e^{-\gamma_1 t}$)

Plug Ansatz for $C_i(t)$ into EoM for $C_n(t)$, $n \neq i$

$$\dot{C}_n(t) = -\frac{i}{\hbar} V_{ni}\, e^{i\omega_{ni}t}\, e^{-\gamma/2\,t}$$

Consistency condition: plug solution for $C_n(t)$ into

$$\dot{C}_i(t) = -\frac{\gamma}{2} e^{-\gamma/2\,t} = -\frac{i}{\hbar}\left(V_{ii}\, e^{-\gamma/2\,t} + \sum_{n\neq i} V_{in}\, C_n(t)\, e^{-i\omega_{ni}t}\right)$$

$\Rightarrow$ fixes γ.

solve for $C_n(t)$

$$C_n(t) = V_{ni}\,\frac{e^{-i(\omega_{in}-i\gamma/2)t}-1}{\hbar(\omega_{in}-i\gamma/2)} \qquad (*)$$

$$\Rightarrow \left(-\frac{\gamma}{2}+\frac{i}{\hbar}V_{ii}\right)e^{-\gamma/2\,t} = -\frac{i}{\hbar}\sum_{n\neq i}|V_{ni}|^2\,\frac{\left[e^{-\gamma/2\,t}-e^{-i\omega_{ni}t}\right]}{\hbar(\omega_{in}-i\gamma/2)}$$

$$\Rightarrow \quad \gamma = \frac{2i}{\hbar}\left[V_{ii}+\sum_{n\neq i}|V_{ni}|^2\,\frac{\left[1-e^{i(\omega_{in}-i\gamma/2)t}\right]}{\hbar(\omega_{in}-i\gamma/2)}\right]$$

V_{ii} just shifts energy (γ_2)

same as 1st order time - independent pert. theory - drop henceforth

Consider $|i\rangle$ an unstable atomic state
decays $|i\rangle \rightarrow |n\rangle = |f\rangle +$ photon w/ energy $\varepsilon = \hbar\omega_{if}$

$$\gamma = \frac{2i}{\hbar}\int |V_{ni}|^2\rho(\varepsilon)\,d\varepsilon\,\frac{\left[1-e^{i/\hbar[E_{if}-\varepsilon-\cancel{i\hbar\gamma/2}]t}\right]}{E_{if}-\varepsilon-\cancel{i\hbar\gamma/2}}$$

~~impossible to~~ can't solve exactly
Assume γ small, drop on RHS

$$\frac{1-e^{i/\hbar[E_{if}-\varepsilon]}}{E_{if}-\varepsilon} = \underbrace{\frac{1-\cos\frac{1}{\hbar}(E_{if}-\varepsilon)t}{E_{if}-\varepsilon}}_{\text{contributes to }\gamma_2} - \underbrace{\frac{i\sin\frac{1}{\hbar}(E_{if}-\varepsilon)t}{E_{if}-\varepsilon}}_{\text{contributes to }\gamma_1}$$

Contribution to γ_2: energy shift from coupling to radiation field

e.g., Lamb shift — separates $2^2S_{1/2}, 2^2P_{1/2}$.

Problematic — apparently divergent,
but can be sensibly calculated, get
finite answer (Bethe: nonrel.,
1040 MHz Weisskopf, Schwinger, Feynman rel.)

Contribution to γ_1:

as $t \to \infty$ $\lim_{\alpha\to\infty} \frac{\sin \alpha x}{x} = \pi\,\delta(x)$

$$\gamma_1 = \frac{2\pi}{\hbar}\int |V_{ni}|^2 \rho(\varepsilon)\, d\varepsilon\, \delta(E_{if} - \varepsilon)$$

$$= \frac{2\pi}{\hbar}|V_{ni}|^2 \rho(E_{if}) = \omega_{i\to n}$$

So, as expected, γ_1 is transition prob. per unit time

Natural line width

Back to (*)

Transition probability to state $|n\rangle$

$$dp = |V_{ni}|^2 \left| \frac{e^{-i(\omega_{in} - i\gamma/2)t} - 1}{\hbar(\omega_{in} - i\gamma/2)} \right|^2 \rho(\omega)\, d\omega$$

$$e^{-i(\omega_{in} - i\delta/2)t} - 1 = \left(e^{-\delta/2 t}\cos\omega_{in}t - 1\right) - i e^{-\delta/2 t}\sin\omega_{in}t$$

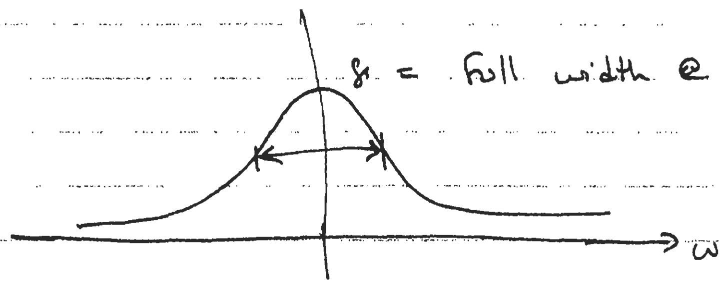
$$\Rightarrow dp = |V_{ni}|^2 \rho(\omega) \frac{1 - 2e^{-\delta/2 t}\cos\omega_{in}t + e^{-\delta t}}{\hbar^2\left[(\omega_{if} - \omega)^2 + \delta^2/4\right]} d\omega$$

δ_1 ; δ_2 just shifts ω_{if}.

as $t \to \infty$,

$$\longrightarrow \frac{1}{\hbar^2} |V_{ni}|^2 \rho(\omega)\, d\omega \frac{1}{\left[(\omega_{if} - \omega)^2 + \delta^2/4\right]}$$

so photon frequency has distribution



state i does not have sharp energy E_i, but natural width

$$\Gamma = \hbar\delta_1 = \hbar A.$$

A is spontaneous emission decay coefficient.

Ex. $A_{(2p \to 1s)} = 6.25 \times 10^8 \ s^{-1}$

$\Gamma/\hbar = 6.25 \times 10^8$ rad/sec $= 100$ MHz is width of 2p state

[Fine structure : 10400 MHz
Hyperfine structure : 1420 MHz]