7.6 N > 2 identical particles & the symmetric group

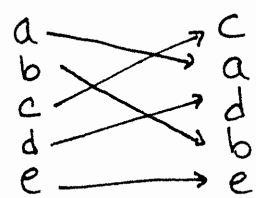
For understanding systems of many identical particles, symmetric group S_N of permutations on N elements is an essential tool.

Permutation group S_N

Given N ordered objects a, b, c, ... a permutation is a general rearrangement of the objects' ordering

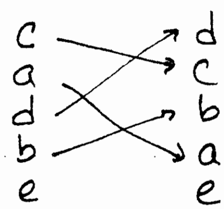
ex.

P :



action of P depends on positions of objects, not labels

P :



Can describe any permutation by cycle structure

$$(1 \leftarrow 3 \leftarrow 4 \leftarrow 2) \quad (5)$$

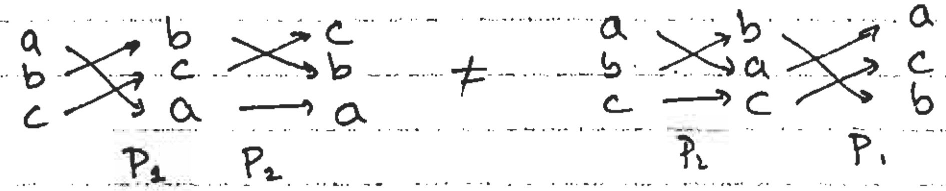
write (1342)(5)

[often drop cycles of length 1 ⇒ (1342)]

N! permutations on N objects form group S_N

S_N is a nonabelian group, $P_1 P_2 \neq P_2 P_1$ in general

ex. $P_1 = (1\,2\,3)$ $P_2 = (1\,2)$



Transpositions $P_{(ij)}$ switch i, j $(i\,j)$.

All permutations can be written as a product of $P_{(ij)}$'s.

Parity of a permutation $\delta_P = (-1)^k$ where k = # of transpositions needed to make P.

Representation theory of S_N

Consider $N!$ - dimensional vector space spanned by all permutations of $\{1, \ldots, N\}$

ex. for $N=3$, $|123\rangle, |132\rangle, |231\rangle, |213\rangle, |312\rangle, |321\rangle$

Any permutation acts on this basis as perm. matrix (one 1 in each row, column, other entries = 0)

ex. $P_{(23)} \rightarrow \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}$ $\begin{matrix} |123\rangle \\ |132\rangle \\ |231\rangle \\ |213\rangle \\ |312\rangle \\ |321\rangle \end{matrix}$

This is regular representation. Contains all irreps.

Young diagrams

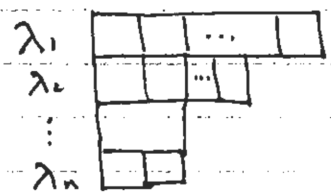
Partition of N : $\lambda_1 + \ldots + \lambda_n = N$

$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$

partitions of N $\overset{1-1}{\longleftrightarrow}$ conjugacy classes $g \sim h^{-1} g h$ in S_N

(cycle lengths)

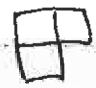
For each partition of N, $\exists$ Young diagram Y_λ



Ex. $N = 2$ $\quad \lambda = (2)$ 

$\lambda = (1,1)$ 

$N = 3$ $\quad \lambda = (3)$ 

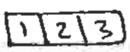
$\lambda = (2,1)$ 

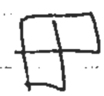
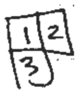
$\lambda = (1,1,1)$ 

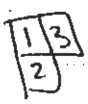
Young tableaux

Given a Young diagram, label with integers 1, 2, ... N.

"standard tableau" : rows & columns increase right & down.

Ex.  $\longrightarrow$ 

 $\longrightarrow$ 

& 

PS 7 due Mon
8 due next wed

~~Symmetric Group~~

Many particle systems

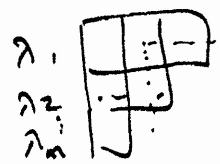
Permutation group S_N

$N!$ permutations on N elements - nonabelian group.

Regular rep.

$$P_{(23)} \rightarrow \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix} \begin{matrix} |123\rangle \\ |132\rangle \\ |231\rangle \\ |213\rangle \\ |312\rangle \\ |321\rangle \end{matrix}$$

Young diagrams $\lambda = (\lambda_1, \ldots, \lambda_n)$
$\sum \lambda_i = N$
$\lambda_i \geq \lambda_{i+1}$



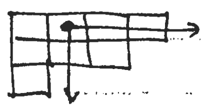
"standard tableaux"
→ ↓ increasing

$$D_\lambda = \frac{N!}{\prod h(i,j)} \quad = \quad \# \text{ of ST / diagram}$$

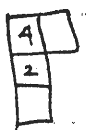
\# of standard tableaux for a diagram:

$$D_\lambda = \frac{N!}{\prod_{\text{boxes}} h(i,j)}$$

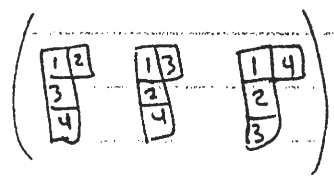
$h(i,j)$ = "hook length" = # of boxes intercepted by lines right & down

e.g.  $h(1,2) = 4$

Ex. $\lambda = (2, 1^2)$



$$D_\lambda = \frac{4!}{4 \cdot 2} = 3$$



<u>Irreps of S_N:</u>

Each irrep. of S_N corresponds to a Young diagram.

D_λ = dimensionality of rep.

also = # of times rep. appears in regular rep.

$$\Rightarrow N! = \sum_{\lambda \, (N \text{ boxes})} D_\lambda^2 \qquad \text{(theorem)}$$

<u>Constructing S_N irreps explicitly</u>

Given a diagram $Y\lambda$, construct a rep. as follows:

for each "standard tableau,"

take linear combination of states — symmetrize on rows,
then antisymmetrize on columns
(using positions)
(can also do consistently w/ labels)

Ex. $N = 3$ $\lambda = (2,1)$ $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \Rightarrow |123\rangle + |213\rangle - |321\rangle - |312\rangle$ (A)

$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \Rightarrow |132\rangle + |231\rangle - |312\rangle - |321\rangle$ (B)

form a basis for a 2D rep. of S_3

Check:

$(123)\,A = |231\rangle + |132\rangle - |213\rangle - |123\rangle = B - A$

$(12)\,A = |213\rangle + |123\rangle - |231\rangle - |132\rangle = A - B$

⋮

Irreps of S_3

$\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array}$	symmetric	$D = 1$	(x1)	1
$\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$	mixed	$D = 2$	(x2)	4
$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$	antisymmetric	$D = 1$	(x1)	1
				$\overline{6} = 3!$

Bases for reps

	$\|123\rangle$	$\|132\rangle$	$\|231\rangle$	$\|213\rangle$	$\|312\rangle$	$\|321\rangle$
$\psi_S = \frac{1}{\sqrt{6}}$	1	1	1	1	1	1
$\psi_A = \frac{1}{\sqrt{6}}$	1	-1	1	-1	1	-1
1: $\psi_{M1,1} = \frac{1}{2}$	1	0	0	1	-1	-1
1: $\psi_{M1,2} = \frac{1}{2\sqrt{3}}$	-1	2	2	-1	-1	-1
2: $\psi_{M2,1} = \frac{1}{2}$	1	0	0	-1	-1	1
2: $\psi_{M2,2} = \frac{1}{2\sqrt{3}}$	1	2	-2	-1	1	-1

Can similarly construct reps of any S_N.

Note: ψ_{M1} symm. under exchanging 1,2 labels
ψ_{M2} antisymm. " " " "

So — Young diagrams label irreps of S_N.
standard tableaux give basis for irreps

Applications of Young diagrams:
A) characterizing & constructing irreps of S_N
B) characterizing multi-particle states in $(\mathcal{H}_k)^N$ under S_N
C) characterizing irreps of $SU(k)$ & constructing on $(\mathcal{H}_k)^N$.
(these 3 conflated in book)

B) Multi-particle states under S_N

Consider N particles each with Hilbert space $\mathcal{H}_k$ of dimension k.

Total Hilbert space: $\mathcal{H} = (\mathcal{H}_k)^N$, $\dim \mathcal{H} = k^N$.

(e.g. $k=2$, spin-1/2 particles; basis $|\pm \pm \cdots \pm\rangle$)

How does $(\mathcal{H}_k)^N$ decompose into S_N irreps?

Answer: for each Young diagram, get 1 copy of irrep for each "standard k-tableau" (nonstandard notation — often "standard" used for this also) satisfying:

- entries $\leq k$
- rows are nondecreasing
- columns are increasing

dim of irrep is still D_λ, of course.

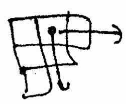
Denote D^k_λ = # of standard k-tableaux for Y.D. λ

Formulae for D^k_λ

writing $\delta_i = \lambda_i - \lambda_{i+1} \qquad i = 1, \dots, k-1$

$$
\begin{aligned}
D^k_\lambda = {} & (1+\delta_1)(1+\delta_2)\cdots(1+\delta_{k-1}) \\
& \times\left(1+\frac{\delta_1+\delta_2}{2}\right)\left(1+\frac{\delta_2+\delta_3}{2}\right)\cdots\left(1+\frac{\delta_{k-2}+\delta_{k-1}}{2}\right) \\
& \times\left(1+\frac{\delta_1+\delta_2+\delta_3}{3}\right)\cdots\left(1+\frac{\delta_{k-3}+\delta_{k-2}+\delta_{k-1}}{3}\right) \\
& \times\cdots \\
& \times\left(1+\frac{\delta_1+\cdots+\delta_{k-1}}{r-1}\right)
\end{aligned}
$$

Alternative expression:

recall "hook length" $h(i,j)$ 

also define $D(i,j) = j - i$ = (column #) − (row #)

0	1	2	...
-1	0	1	
-2	-1	0	
⋮			

$$D^k_\lambda = \prod_{\text{boxes}} \frac{(k + D(i,j))}{h(i,j)}$$

equivalent to above.

Theorem: $\sum D^k_\lambda D_\lambda = k^N$

Ex. 3 spin-1/2 particles : 8D Hilbert space

irreps:

[−|−|−]
[−|−|+]
[−|+|+]
[+|+|+]

} $D_\lambda = 1$ symmetric states

$\left(D^2_\lambda = \frac{2}{3} \cdot \frac{3}{2} \cdot \frac{4}{1} = 4 \right)$ (1+3) [$\delta_1 = 3$] [h = [3|2|1], D = [0|1|2]]

~~[ ] / [ ] / [ ]~~

[−|−] / [+]
[−|+] / [+]

} $D_\lambda = 2$ mixed states

$D^2_\lambda = (1+1) = 2$ [$\delta_1 = 1$]

$$1 \times 4 + 2 \times 2 = 8$$

To get states, plug into states for standard tableaux
– get redundancy ; linear dependencies or vanishing

explicitly, : [−|−] / [+]

$$\psi_{M_1,1} = \frac{1}{\sqrt{2}} \left(|{-}{-}{+}\rangle - |{+}{-}{-}\rangle \right)$$

$$\psi_{M_1,2} = \frac{1}{\sqrt{6}} \left(|{-}{-}{+}\rangle + |{+}{-}{-}\rangle - 2|{-}{+}{-}\rangle \right)$$

$$\psi_{M_2,i} = 0 .$$

We now understand: • irreps of S_N, dim D_λ
• how to decompose regular rep & $V(\mathcal{H}_k)^N$ into S_N irreps.
(including multiplicities D_λ, D^k_λ & explicit wt's)

c) Classify irreps of SU(k)

Last semester, classified irreps of su(2):
for each $j \in \mathbb{Z}/2$, $\{|j,m\rangle, m = -j, \dots, j\}$

Fundamental rep. of SU(k): k-dimensional defining rep. on $\mathcal{H}_k$.
Denote by □

Irreps found by considering action on $(\mathcal{H}_k)^N$, decomposing.
irreps determined by S_N symmetries — action of SU(k) leaves symmetry structure fixed since $[SU(k), S_N] = 0$.

<u>Theorem</u>: irreps of SU(k) $\overset{1-1}{\longleftrightarrow}$ Young diagrams with $\leq k$ rows

Dim of irrep λ = D^k_λ

# of times λ appears in $(\mathcal{H}_k)^N = D_\lambda$
(proof later)
[include k rows; Y_λ w/ k rows $\sim Y_\lambda$ w/ $<k$]

<u>Comments</u>:

- Fits with $\sum_\lambda D^k_\lambda D_\lambda = k^N$
- Explicit rep. found by action of SU(k) on states associated with standard k-tableaux.
- Columns w/ k boxes → totally antisymmetric, act as singlet & can be dropped.

Ex. SU(2) reps

□ Fundamental ($j = 1/2$) $D^2_\lambda = 2$

□□ ($j = 1$) $D^2_\lambda = 3$

Young diagrams:

A) reps of S_N irreps $\overset{1-1}{\longleftrightarrow}$ Y.D.'s.

in N! - dim reg. rep.

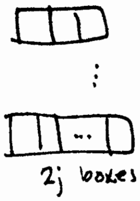
$$\dim = D_\lambda = \#\text{ standard tableaux}$$
$$\text{mult} = D_\lambda$$
$$N! = \sum_\lambda D_\lambda^2$$

B) S_N reps in $(\mathcal{H}_k)^N$

$$\text{mult} = D^k_\lambda = \#\text{ standard } k\text{-tabl.}$$
$$k^N = \sum D_\lambda D^k_\lambda$$

C) SU(k) reps in $(\mathcal{H}_k)^N$

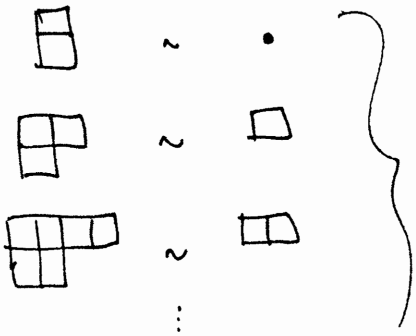
$$\dim = D^k_\lambda$$
$$\text{mult} = D_\lambda$$



$(j = 3/2)$ $\quad D^2_\lambda = 4$

$(j = \text{anything})$ $\quad D^2_\lambda = 2j+1$

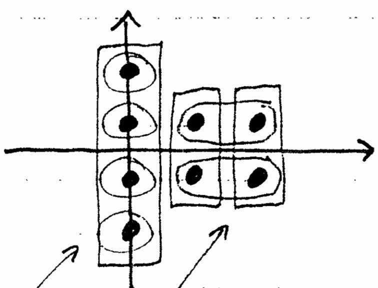
also:



appear in $(\mathcal{H}_2)^N$, needed for countings but D^k_λ same for different equivalent diagrams.

Example: Decomposition of $(\mathcal{H}_2)^3 = 8d$ space under $SU(2)$, S_3

(3 spin-1/2 particles $\Rightarrow$ $(j = 3/2) \times 1$, $(j = 1/2) \times 2$



$\square$ = $SU(2)$ reps

$\bigcirc$ = S_3 reps

$D_\lambda = 1 \qquad D^2_\lambda = 4$

(4 $D=1$ reps of S_3,
1 $D=4$ rep of $SU(2)$)

$D_\lambda = D^2_\lambda = 2$

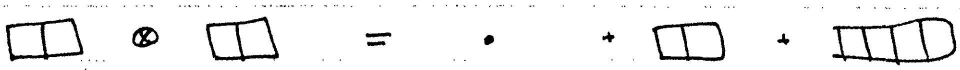
(2 $D=2$ reps. of S_3, $SU(2)$)

Tensor product reps

First do SU(N)

Want decomposition of tensor product in irreps

Ex. (for SU(2))

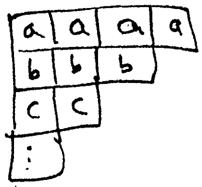


(j=1) ⊗ (j=1) = (j=0) + (j=1) + (j=2)

$3 \times 3 = 1 + 3 + 5$

General rule

1) label second diagram w/ a, b, c ... in 1st, 2nd, 3rd rows...

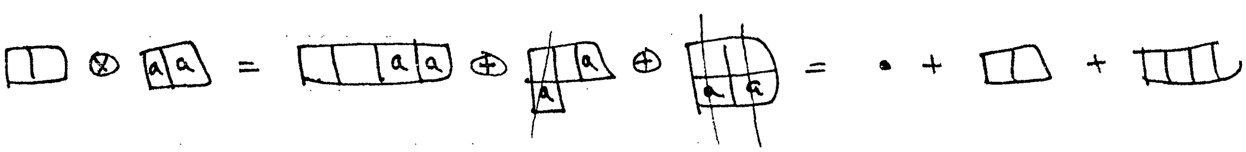


2) attach a's to the 1st diagram in all ways such that
   a) no 2 a's in same column
   b) still a Young diagram (row lengths nonincreasing, etc...)

   repeat with b's, c's, ...

3) read letters in right-left order, rows from top down to get string aaba...
   reject if to left of any symbol more b's than a's, c's than b's, etc...

Ex. for SU(2)



Note that decomposition of $(\mathcal{H}_k)^N$ is just $\underbrace{\square \otimes \square \otimes \cdots \otimes \square}_{N}$

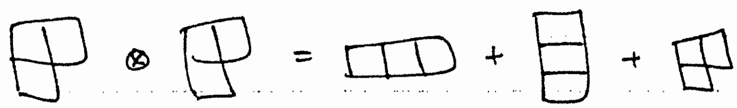
repeating rule, adding 1 box @ a time gives all standard Young tableaux with $\leq k$ rows
(labeling = order of placement of boxes)

$\Rightarrow$ proves D_λ = <u>\# of</u> times Y_λ appears in $(\mathcal{H}_k)^N$

Would like analogous formula for tensor product of S_N representations, giving decomp. of $Y_\lambda \otimes Y_{\lambda'}$ in S_N irreps.

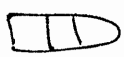
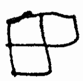
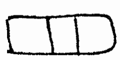
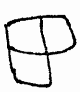
No simple algorithm known for general case!

Special cases:  $\otimes Y = Y$



2 ⊗ 2 = 1 + 1 + 2

Can show from following argument:

		\# SU(2) reps (D_λ)	\# S_3 reps (D^2_λ)	
$(\mathcal{H}_2)^3 \Rightarrow$		1	4	$(1\cdot 4 + 2\cdot 2 = 8)$
		2	2	
$(\mathcal{H}_4)^3 \Rightarrow$		1	20	$(1\cdot 20 + 2\cdot 20 + 1\cdot 4) = 64)$
		2	20	
		1	4	

Since $\mathcal{H}_4 = \mathcal{H}_2 \otimes \mathcal{H}_2$, we must have for S_3 reps:

$$(4\,[3] + 2\,[2,1]) \otimes (4\,[3] + 2\,[2,1])$$

$$= 16\,[3] \oplus 16\,[2,1] \oplus 4\,([2,1] \otimes [2,1])$$

$$= 20\,[3] \oplus 20\,[2,1] \oplus 4\,[1,1,1]$$

$$\Rightarrow [2,1] \otimes [2,1] = [3] \oplus [1,1,1] \oplus [2,1].$$

Can do more explicitly with states — $\psi_S = \psi_M \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{\Phi}_M$
$\psi_A = \psi_M \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tilde{\Phi}_M$

$$\psi_S(r_1, r_2, r_3; s_1, s_2, s_3) = \psi_{M,1}(r_1, r_2, r_3)\,\psi_{M,1}(s_1, s_2, s_3) + \psi_{M,2}(r_1, r_2, r_3)\,\psi_{M,2}(s_1, s_2, s_3)$$

$$\psi_A(r_1, r_2, r_3; s_1, s_2, s_3) = \psi_{M,1}(r_1, r_2, r_3)\,\psi_{M,2}(s_1, s_2, s_3) - \psi_{M,2}(r_1, r_2, r_3)\,\psi_{M,1}(s_1, s_2, s_3)$$

General result: Antisymmetric rep only appears in $Y \otimes \tilde{Y}$, $\tilde{Y} = \text{transpose}(Y)$ [Natural: (A)(S) (A/S), 1 2 2 1]

ex. $[3,1] \otimes [2,1,1] = [1,1,1,1] + [2,2] + [2,1,1] + [3,1]$

Example applications:

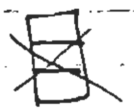
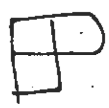
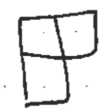
1) $(2p)^3$ states in Nitrogen (see also Sakurai)

total # of states: $\binom{6}{3} = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$

Combined space-spin wavefunction must be antisymmetric.
write in basis of $\Psi_{space} \otimes \Psi_{spin}$ space: $j=1$ ($\mathcal{H}_3$)
spin: $j = 1/2$ ($\mathcal{H}_2$)

Need to get  in tensor product of $Y_{space} \otimes Y_{spin}$

Possibilities:

space	spin	
	 $D^2_\lambda = 0$	
 $D^3_\lambda = 1$	 $D^2_\lambda = 4$	(4 states) $\ell = 0,\ s = 3/2 \Rightarrow {}^4S_{3/2}$
 $D^3_\lambda = 8$	 $D^2_\lambda = 2$	(16 states) $\ell = 2, 1,\ s = 1/2 \Rightarrow {}^2D_{5/2},\ {}^2D_{3/2},\ {}^2P_{3/2},\ {}^2P_{1/2}$

(Note: (8) of SU(3) contains (4) + 2×(2) of SU(2)
)

Example: construct ${}^2D_{5/2}$ $m = 5/2$ state

must have $\psi_M(++0)$ space
$\psi_M(\uparrow\uparrow\downarrow)$ spin

$$\psi_A(++0;\ \uparrow\uparrow\downarrow) = \psi_{M,1}(++0)\,\psi_{M,2}(\uparrow\uparrow\downarrow) - \psi_{M,2}(++0)\,\psi_{M,1}(\uparrow\uparrow\downarrow)$$

$$= \frac{1}{\sqrt{6}}\Big[\,|+^\uparrow +^\downarrow 0^\uparrow\rangle - |+^\downarrow +^\uparrow 0^\uparrow\rangle + |+^\downarrow 0^\uparrow +^\uparrow\rangle - |+^\uparrow 0^\uparrow +^\downarrow\rangle + |0^\uparrow +^\uparrow +^\downarrow\rangle - |0^\uparrow +^\downarrow +^\uparrow\rangle\Big)$$

Note: can write any antisymmetric state in Slater determinant form

$$\psi_A = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_1(x_1) & \phi_1(x_2) & \cdots & \phi_1(x_N) \\ \phi_2(x_1) & \phi_2(x_2) & \cdots & \phi_2(x_N) \\ \vdots & \vdots & & \vdots \\ \phi_N(x_1) & \phi_N(x_2) & \cdots & \phi_N(x_N) \end{vmatrix}$$

[obvious generalization to include spin, etc...]

State $\psi_A(++0;\uparrow\uparrow\downarrow)$ uniquely determined by this form.

- Not true for other states (e.g. $^2P_{3/2}$, $m = 3/2$, [HW])
  [can fix either by using tensor product formalism or operator manipulations]

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2) Quarks in a baryon

quarks have wavefunction in $\mathcal{H}_{space} \otimes \mathcal{H}_{spin} \otimes \mathcal{H}_{flav} \otimes \mathcal{H}_{color}$ $(SU(3))$

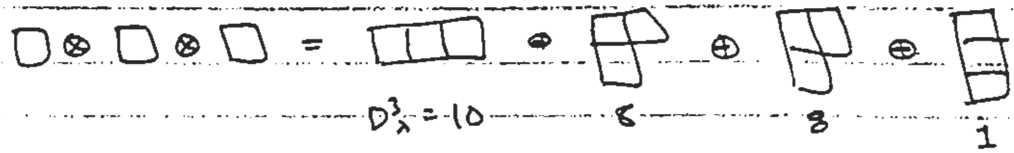
Consider 3 light quarks: u, d, s

Live in SU(3) flavor multiplets: q in □ (3), $\bar{q}$ in ⊟ $(\bar{3})$

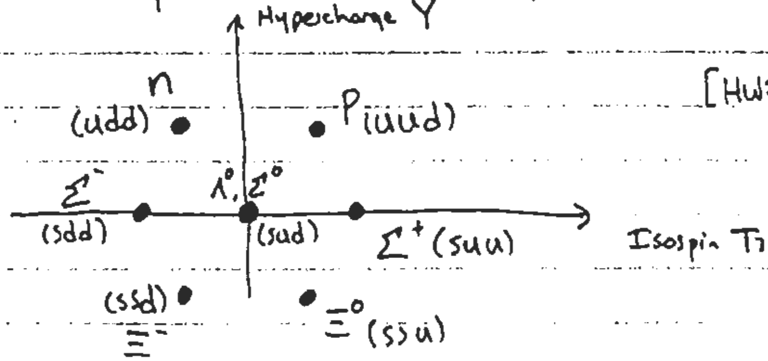
mesons $(q\bar{q})$:

(as SU(3) reps) $3 \times \bar{3}$ = $D^3_\lambda = 8$ (octets) + $D^3_\lambda = 1$ (singlets)

baryons:
(qqq)

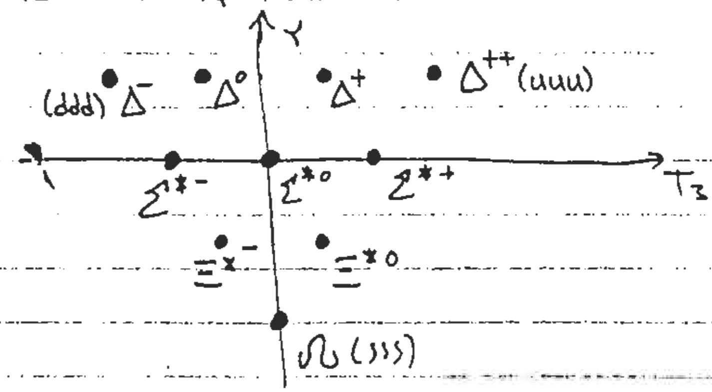


spin $1/2^+$ baryon octet



[HW: wfunction of proton]

spin $3/2^+$ decuplet (□□□)



early puzzle:

baryon decuplet $\Delta^{++}, \ldots$ □□□ flavor

have $S = 3/2$ (□□□ spin)

in ground state of spatial wf ⇒ □□□ space

where is antisymmetry?

answer: (column of 3 boxes) in color $\psi_{color} = \frac{1}{\sqrt{6}}\left(|RBY\rangle - |RYB\rangle + \ldots \right)$

Refs. on group theory & Applications to QM:

"A course on the Application of group theory to QM", Irene V. Shensted

"Group theory", M. Hammermesh