[SQUEAKING]

[RUSTLING]

[CLICKING]

PROFESSOR: OK, good. So last time, we quantized this theory. So let me just write down the theory again. So we quantized this theory. And so let me also just write down its canonical momentum, density, and then the Hamiltonian density would be 1/2 pi square.

And the classical equation of motion-- and so we're not repeat the quantization procedure. And we wrote down the most general solution at the level-- can be written as the following. This factor is the convention. And then we have a k u k plus a k.

So from now on, I will suppress the hat. So I will suppress hat. You should always now view it as a quantum field-- quantum operators. And the plus u k star. And the u k is the complete set of solution, which is given by exponential minus i omega t plus i k x.

So this is just a basis of solution. And so this is the most general-- so this is the complete solution to the operator equation for this operator phi. And then you can also find its-- find its conjugate momentum density.

So the pi, you can just take the derivative straightforward. I will not write it explicitly. And then we discussed that you can impose a canonical commutation relation which is that the phi t x phi t x prime equal to pi t x, and pi t x prime should be 0. And then phi t x pi t x prime-- then given by the delta function.

So that's what we did at the end of last lecture. And then you can just plug it in-- plug this-- plug those expression into here. Then you can find the commutation relations between those integrations. So a k and a k dagger integrating constant of your operator equations. They are constant operators.

And so when you plug them in, then you find the canonical-- then you find the commutation relation between a k and a k dagger. And then you find ak and a k prime equal to a dagger k and a dagger k prime is equal to 0 and a k a k prime dagger is equal to delta function in the wave space.

So now, if you look at those expressions-- so they essentially look-- they essentially-- we essentially find the infinite number of harmonic oscillators. And each harmonic oscillator labeled by continuous-- by a set of continuous number k. And then the commutation relation is just like a continuum generalization of standard a a dagger equal to 1.

And they are independent because of this delta function. And if k is not equal to k prime, and then you get 0. They commute. So you can see the story a little bit sharper. That indeed, we just get the infinite number of harmonic oscillators.

So these are just commutation relations. This still does not tell you we have a harmonic oscillator. To see harmonic oscillator, we actually need to calculate the Hamiltonian. So now let's try to calculate the Hamiltonian. So now we can try to calculate the Hamiltonian.

And so this is just-- can just do it-- straightforwardly just plug that in-- just plot it plug that expression in we wrote above. And then you just plug-in the expression for the phi and the pi into that equation, and then just straightforward to calculate it. And I will not go through the details. Of Yeah, it's a straightforward calculation.

So then you find maybe after some minutes-- then you find that the following answer. So now, you-- when you plug them in, then you find that you can do actually do the spatial integral because you just have the plane wave. You can just do the spatial integral. And when you do the spatial integral, then that gives you a delta function momentum space, et cetera. And then you can reduce everything into a single moment-- a single k integral.

And then what you find is that you get omega k. And then you find a k dagger a k. And then plus a k dagger. So let me see. Yeah, actually-- yeah, I think 1/2. Just get that.

So this is the standard expression for harmonic oscillator now. Actually, it's exactly identical to just some-- yeah, integral is like sum-- just sum over continuum harmonic oscillators. And each with frequency omega k, and each harmonic oscillator is labeled by this number k.

And so we can do from the standard trick to write it actually as to write as a commutator. So we can just write it as-- and then we just have this. So we combine these two terms together and introduce a commutator. You just have a dagger a and then plus the commutator between them.

So in the standard story, this is equal to 1. Then you get the standard answer that you get omega a dagger a plus 1/2. But here, we can again-- let me just write one more step. So this is k omega k a k dagger a k, then plus E0. And now E0 is the sum of all the zero point energy of all the harmonic oscillators.

So 1/2 omega k-- except that the tricky thing is the following. Except now, we have this commutator of a k with a k dagger. In the standard case, this is equal to 1. But in our case, when you set k prime equal to k, you get the delta 0. So we get delta 0.

So then you get that 2 pi cubed delta 3 0. And remember, this 0 is in momentum space. And so this is in k space, not in the-- we have two kinds of delta functions in the-- delta function in k space, delta function in coordinate space.

So here, I put the subscript k-- here means that this is the delta function 0 in k space. And so this is the zero point energy. So this zero point energy is divergent, but we will comment on that a little bit later. Do you have any questions on this?

Good. So remark-- let me just make a couple of remarks. First is that-- so something interesting happened. So when you plug-- when you look at the expression for phi and the pi, both phi and the pi depend on time explicitly. So there's a time dependence here.

And here, we only integrate over spatial direction. So in terms of integration, we don't do anything with time. But what do you notice here? What do you notice there?

STUDENT: So you drop the time dependence?

PROFESSOR: Yes, it's not to say we drop time dependence because time dependence disappeared. Do you know why?

STUDENT: I mean, the Hamiltonian doesn't explicitly depend on time.

PROFESSOR: No, the Hamiltonian-- phi is time dependent, pi is time dependent. If you plug-- if you plug those expression into here, certainly this-- certainly inside there is depend on time explicitly. This guy is depend on time explicitly.

But in the end, after you do the integration over spatial directions, then you find in the end the time dependence actually cancel during the calculation. So do you have a guess why somehow this should cancel? Yes?

STUDENT: Energy conservation or something?

PROFESSOR: Yeah, exactly. So you have in your pset that the time translation-- so this system is time translation symmetric. It means that you should have the-- it means that the H is a conserved number. It's a conserved quantity. And by-- so indeed, you see that this is independent of time. This is a conserved quantity.

We see it explicitly. So there's no time dependence since H is a conserved quantity. So this integration is absolutely crucial, because if you don't do integration, Hamiltonian density is not-- it's only the total Hamiltonian is conserved. So this is a comment.

And the second -- so since I only have two comments, let me just call it one rather than call it 0. So for the second comment is that-- yeah, just repeat. We already said-- you just have OFT of phi.

And we see this QFT of phi reduce this to essentially a system of a continuum of harmonic oscillators labeled by k with frequency omega k. So we call this omega k is defined to be k squared plus m square.

So as we mentioned last time, the fact that we actually see infinite number of harmonic oscillator is actually not surprising from the perspective that this field theory can be written as the continuum limit of a chain-- say, of atoms connected by springs. So when you have a chain, we wrote down a one-dimensional system. You can easily generalize to three-dimensional.

Yeah, so you have those chains of atoms connected by spring, and they are just coupled oscillators. And then you can just diagonalize them, and find the normal spectrum, and each of them is just a harmonic oscillator. And so when we find those solution by doing Fourier transform-- and essentially, we are just diagonalize those interactions between those springs.

And yeah, so that's why we get infinite harmonic oscillator. So from that way, if you think about it, it's totally unsurprising we get the infinite number of harmonic oscillators. But it does surprising-- it is surprising if you think about it from the point of view of a field theory. So if you find when you quantize a field theory, and then you find in the end you get a bunch of harmonic oscillator.

And before doing that, it's hard to-- before you actually carry out the quantization, it's actually hard to anticipate that if you don't have this intuition from this discrete system. Do you have any questions on this? Yes?

STUDENT: So when you say that there's an infinite amount of harmonic oscillators, are you saying that there's an infinite amount at any given point in space, or that each given point in a continuum of location there is one harmonic oscillator?

PROFESSOR: Yeah, that's a very good question. And so it's at each point, you have a harmonic oscillator. Yeah, at each point-- essentially, at each point you have a harmonic oscillator, and then they connected in space. Yes?

STUDENT: Is there any reason why you didn't compute the integral in E0? You have a delta function in there?

PROFESSOR: Sorry, say it again?

STUDENT: So the integral if E0 equals. Why didn't you just--

PROFESSOR: Oh, this-- we cannot do the integral.

STUDENT: No, up there-- the zero point energy.

PROFESSOR: Oh, right. Yeah, we will talk a little bit later, because later, we are going to elaborate on this a little bit later because we are going to try to give interpretation of this. Yes?

STUDENT: In homework 1, we found that a hat was time dependent. Is a hat not time dependent here, and is that why the Hamiltonian is not time dependent?

PROFESSOR: Yeah, a hat is a constant operator. Yeah, because a hat is an integration constant of your operator equation. Yes?

STUDENT: Is there an analog of coherent states for each harmonic oscillator in the QFT?

PROFESSOR: Yeah, there's an analog of coherent state. Yeah, indeed.

STUDENT: Does it tell you anything useful if you are in Heisenberg interpretation.

PROFESSOR: Yeah, you can-- yeah, we will-- you will see-- yeah, this question will become clearer when we talk a little bit about the Hilbert space. Other questions? OK, good.

Now, let's-- before we talk about the Hilbert space, we need to talk about two things. So here-- yeah, here, I have a dot. And the second thing-- so Hamiltonian is an important quantity. So this is one of the conserved quantity.

So there are other conserved quantities. So there are other conserved quantities, and one of them is the conserved quantity corresponding to spatial translation. So the spatial translation gives you conserved momentum-- momentum conservation. So from spatial translation, then you get this conserved charge p i.

Again, you should have done this in your homework to have the following form-- d3 x pi. So this is the Noether charge for the conserved quantity associated with the spatial translation. And we interpret it as a spacetime momentum. This should be interpreted as spacetime momentum.

So this is-- so here, we have two momentum here. You don't confuse them. So this pi is the canonical momentum conjugate to this canonical variable-- pi is the canonical momentum conjugate to this field variable phi. And this is the genuine physical spacetime momentum. It's the momentum of your full system-- of your full physical system.

And again, we can just plug it in-- we can just plug in the explicit expression of pi and phi. We already know its time evolution. We already know the time evolution, and then we can plug them in.

And then you find that the answer is given by-- again, after some smaller number of minutes calculation. And then you find that the answer is given by this. In this case, there is no zero point energy. You just have this expression.

And furthermore, you also have a Lorentz transformation-- Lorentz symmetry. And then that will lead to, say, the conserved charge associated with Lorentz transformation. And again, you can find it explicitly. And so in your pset 2, you will find the expression of this M mu nu in terms of phi and pi.

And then you can again express this guy in terms of a and a dagger. But I didn't have the guts to assign as the part of the pset, so I used it as a bonus problem because that calculation-- involved a little bit extra calculation. Yeah, it involves a little bit slightly more tedious calculation, so I used that as a bonus problem for those people who like to have some more fun.

Yeah, anyway, so this you can-- will see in your pset. Good. And then again, pi, you see the explicitly that this is time independent-- it's time independent. Good. And then the next thing is let's talk about this zero point energy.

So this zero point energy, we can write it as the following. So we can-- yeah, so this is a constant. So you can take it outside of this k integral. This does not depend on k because the-- yeah.

So this is just 2 pi cube delta 3 0. So this is-- remember, this is a k space delta function. And then you have the d3 k. So that's the expression we get. So now, let's try to understand what's the meaning of this term?

Let's try to understand the meaning of this term. To understand the meaning of that term, we need to do a little bit of a mathematical trick. We need to a little bit of a mathematical trick. So remember the definition of delta function, we have 2 pi cubed delta 3k equal to the Fourier transform of the exponential of i k x.

And so this is the definition of-- essentially, the definition of the delta function. And now, here, we want to set k equal to 0. So let's set k equal to 0 here. And then we find that these 2 pi cubed theta 0-- so let's remember this again in the k space. And then we just set k equal to 0 here.

And then what is this? If you set k equal to 0, you get what this? Yes?

STUDENT: The volume--

PROFESSOR: Yeah, that's right.

STUDENT: --of the full space.

PROFESSOR: You just get the volume of the full space. Let's just imagine you-- in order to make sense of this quantity, you can put the whole universe in a big box. Imagine the whole universe is in a big box, and then this is the total volume.

And now, since this is the volume, now we have a very good interpretation of this quantity. And now we can write E0 as merely volume times epsilon 0. And then this quantity epsilon 0 then have the interpretation of the energy density. So now this is energy density. So this energy density, we can write it in one more step-- so continue to here.

So we can write it one more step. Plug in the expression value of omega k. So this is the-- just 1/2. Sorry, I forgot 2 pi d k.

And then you have k square. Yeah, so that's what you get. So how do you like this integral?

STUDENT: So just a question to go back here. Wouldn't it be quicker if you just left the delta function inside of your integral and just say that it's one for your entire-- like for all k. So why are those two pictures equivalent? Because in that picture, you don't get the volume?

PROFESSOR: Sorry, say it again? What's two picture?

STUDENT: If you didn't pull out your delta function from your integral-- you just did it with-- that delta function is one for all k.

PROFESSOR: No, because this is just a constant. Just when you have an integral of a constant, you can always pull it out. Because delta 0 does not depend on k anymore. Good. So how do you like this integral? Can you do it?

STUDENT: No, it diverges

PROFESSOR: Good. So it's fruitless to do this integral because this is divergent. And there's very good reason for this divergence physically. So this is just the energy density. So essentially, this is the energy per unit volume-- say in some unit volume.

So let's imagine you have a unit volume here. Remember, this-- beside-- before this, corresponding to you take a discrete system with some lattice spacing a, and then you take the lattice, and each lattice point, you have a harmonic oscillator, and then you take a lattice spacing-- a equal to 0. So that means for any unit volume, when you take a equal to 0, you have infinite number of oscillators inside.

So essentially, when you-- yeah, you have the lattice of oscillators at each point. Anyway, I will not try to draw it. And then when you take a to 0, you have any unit-- any volume unit, you will have the number of oscillators go to infinity, and that's where this divergence come from.

It just come from-- in field theory, you have a continuum degree of freedom. So at each point, you have a degree of freedom, and within any volume, then you definitely have infinite number of degrees of freedom. so this is just from continuum of freedom.

So this is the first time you see a diverging quantity in quantum field theory, but you will soon find that this is-- soon, you find that this is normal. It will be a fact of life. Just you will see divergence very commonly-- and so-- because you have continuum degrees of freedom. So the whole thing about quantum field theory is to find a way to deal with those divergences.

And one of the key differences between quantum field theory and just finite quantum mechanics or finite number degrees of freedom is because of those divergences. And the big part of quantum field theory is to understand how to treat those divergences. And they actually don't affect your physics, but you do have to develop sometimes sophisticated tricks to treat OK,

Good. And this infinite answer is also closely connected to a very famous problem you may have heard. It's called the cosmological constant problem. Because this tells you that any quantum field theory have an infinite, say, 0 point energy.

And so you may say, OK-- so infinite-- whenever we say something infinite, there's one thing you always do to treat it. Can you guess what is the thing you always do when you see infinities? Yes?

STUDENT: Just like subtract infinity.

PROFESSOR: Yeah, that's one idea. That's one idea, but to subtract infinity is very hard. In your calculus class, when you subtract infinity from infinity, you can get infinity, so you have to be very careful. Yes?

STUDENT: Just ignore the term.

PROFESSOR: That's a lot of very good idea.

[LAUGHTER]

Indeed, that's what we often do. Yes?

STUDENT: Divide by.

PROFESSOR: That's also-- indeed. But to do all those things, you have to do one thing first. Yes?

STUDENT: You might approximate it, it's like 1 over the Planck constant.

PROFESSOR: Yeah, it's also very close. You need to do--

STUDENT: [INAUDIBLE]

PROFESSOR: Hmm?

STUDENT: [INAUDIBLE]

PROFESSOR: Yeah, exactly. You need to find a way to make it finite first, and then you can subtract it. And then you can-- yeah, just like when you sum 1 plus 1/2 plus 1/3, et cetera. You get an infinite series.

But if you want to estimate the outcome-- yeah, you get divergence. But you always try to cut it off the series, and then approximate definite answer. And here is similar. So here we always put some momentum cut-off. So imagine the momentum is smaller than some momentum-- say, some value lambda. And lambda corresponding to maybe to the scale which this quantum field is no longer apply, because nobody told you that this quantum field theory should apply at all length scales.

Because for example, in this lattice model, this scale will be 1 over the lattice spacing. Anyway, so once you cut it off, still you get a pretty big number. And you cannot ignore it because this is a physical zero point energy. In principle, you can measure it.

But in real life, we don't see it. So we have quantum fields flying around all the time, but we don't see this big vacuum energy. So this is called the cosmological constant problem.

Actually, there was just a colloquium last week about this cosmological related to this cosmological constant problem. Good. Any questions? Now, we can talk about Hilbert space.

So as a harmonic oscillator-- so we can first define the vacuum state. So now, what we will do is indeed in quantum field theory itself, we can just ignore this E0. From now on, we will just ignore this E0 because this just to give you overall-- just give you a constant, which does not do anything. It's like the potential energy E and M.

Yeah, just like the unit-- yeah, just like the-- anyway, so from now on, we ignore this term. So often, I will just write the Hamiltonian-- I just write this term. But later, we actually see examples-- later, we will see examples. Actually, this E0 can actually have physical implications, but just for our current purpose, we ignore it.

So now, the lowest energy state-- then it's clear from here because this is just a constant. So the lowest energy state ground state, which is we often call the vacuum state, means there's nothing there, is given by a k 0 equal to 0 defined to be for any k. So the ground state satisfy, which I denoted by 0-- then satisfy annihlated by this ak.

And then the general state-- you can just-- so general states have the following form. Say I can write n k 1. So the k 1 oscillator excited n times. n k 2-- n k-- the k 2 oscillator excited n k 2 times, et cetera, which is given by-- which is proportional to a k 1 dagger to the power n k 1 a k 2 dagger n k 2 etcetera acting on 0 just like we have large number of harmonic oscillators. Questions on this? Yes?

STUDENT: How do you know there exists some state that's annihilated by every single annihilation operator?

PROFESSOR: You postulate, but you can actually write it down its wave function. Just as in the standard harmonic oscillator case, you can starting from a dagger annihlated by a. You can start from this equation to write down its wave function. And here, you can write down the wave function for the vacuum state to in terms of phi. Yes?

STUDENT: In the notation for the ket there it looks like we have countable number of frequencies? Is that, like why is that?

PROFESSOR: You mean here?

STUDENT: Yeah.

PROFESSOR: Yeah, here, I'm just saying it depends on which k are excited. Yeah, that's a very good question. I will comment on related issues very soon. But here, I just write down some state. Yeah, excite k1, k2 as I want. I'm not saying that this is the-- yeah, you can have as many as you want.

STUDENT: Yeah, but if you use that notation it's still.

PROFESSOR: That's right. It's true. Yeah, but I write it-- when I write it this way, it implies I only excite countable number of them. But in principle-- yeah, we will soon touch a point, which is related to your question. Other questions?

OK, good. And so for example, the simplest excited state would be just excite one of them. So let me denote this by k. And then the simplest way you just excite two of them.

So this k1 and k2 can be the same. If these k1 k2 are the same, and then just like the square. And here, also I will not be very careful about the normalization. And now, we ask, what are the physical interpretations of those states?

So now let's ask what are the physical interpretations? So for this purpose, we can just look at their quantum numbers under, say, the Hamiltonian and under the spacetime momentum. So for example, H 0-- so if you act on H on 0, then of course, you get 0 assuming we throw E0 away.

Say ignore E0 so far. And then so for the ground state-- and the p i acting on 0. Of course, it's 0. So you can see just because this have ak here. And so E also-- when we throw this away, you have this --yeah.

And now, let's look at the excited state. So now to look at this state. So H acting on k-- so this answer is obvious because all different oscillators, they are independent of each other, and this just gives you omega k. We can just use our result for harmonic oscillator.

And now, you can look at the momentum-- so this is a spacetime momentum operator acting on here. And then it has k i. If you look at this expression, when you do the-- act this on that-- you act this on that, and when you can just use the standard trick when you do the commutator.

And then the particular k for this is chosen, and then the eigenvalue will be just given by that particular k i. So this is one step of calculation there, and you should do it yourself. 1-minute calculation. This a delta function will be generated, and then that will get rid of that integral, and then you will pick up this k i.

So now, this equation has very-- now has an obvious physical interpretation. So that means that this state k has spacetime momentum. So now, I write down a four-momentum omega k n k. So this is exactly the momentum of a relativistic particle on shell.

So this is of a relativistic-- so this is a momentum of a relativistic particle, because omega k is equal to k squared plus m square of mass m. Yeah, so this means that the p squared. So this is a four-vector squared is equal to m-- yeah, so this satisfies the p squared equal to minus m square.

So we can just-- so it's very logical just to interpret this as a particle of mass m, because we can just interpret it as a particle of mass m. Good? So now, let's look at this one. So again, the calculation is very simple.

So you find for that one-- so you find for that one-- for k1 k2, you find the energy-- energy just defined-- again, you find that this is an energy eigenstate of the H. It's an eigenstate of H with energy eigenvalue omega k1 plus omega k2. And it's a momentum eigenstate of P i with an eigenvalue given by k1 plus k2. So E, k are eigenvalues of H and P.

STUDENT: Question.

PROFESSOR: Yeah?

STUDENT: Yeah, so could that mass be 0?

PROFESSOR: So here-- so this mass is not 0 because omega k is defined to be this-- it's defined by my theory-- defined by my Lagrangian. So this is the parameter of your-- yeah, of your action. Yes?

STUDENT: So I have a -- with the energy spectrum, does the spacing-- does it happen the same way where the spacing gets smaller as you go to higher and higher energies or?

PROFESSOR: Yeah, for here, it's uniform. Here, just for each k, it's uniform. But of course, when you add them together, you get something very complicated. But for each k, you just uniform.

So the very-- so the most natural way to interpret this is just have two particles of momentum of four momenta omega k1 k1 and omega k2 k2. And similarly, you can do this for any state like this.

They are all eigenvectors of H and P. So n k 1, n k 2, you catch-- et cetera.

So this corresponding to n k 1 particles of momentum. They're all on shell-- on shell particle of momentum k1 and n k 2 has two particles of momentum k2. So this tells you one thing.

So now let me just make some remarks. Any questions on this before I make my remarks? Good. So the first point is that now you can see this can describe any number of particles.

So mathematically, in our description, there are harmonic oscillators. But each excitation of the harmonic oscillator is from the spacetime point of view corresponding to a particle. So this is the beautiful thing of this theory.

And then due to-- because of the commutation relation, a k 1 dagger, a k 2 dagger equal to 0 for any k1 k2. So you have full symmetry. So when you construct the state, you can just commute them as you want.

So that means they're-- so full symmetry in permuting all these different particles in the general state. So this tells us these are bosons. And two is that all particles have positive energy. So even though that E squared equal to k squared plus m squared-- this equation have two solutions-- plus minus omega k.

But when you look at physical state-- when you look at your state and look at the eigenvalue of state-- so all particles have physical energies, so you don't have this negative energy problem associated with taking the square root. Also you have total energy of a state is equal to sum of energies of all the particles.

So this tells you there's no interactions between them. Because if you have potential energies between particles, and then that will change the energy. When you put the two particles together, will no longer be the same over the sum of the individual energy for each particle. And so that tells you there's no interactions. So this is a theory of free particles.

Now, this is a good starting point. At least now, we have particles. Questions on this? Yes?

STUDENT: So [INAUDIBLE] a k 1 let's say, and you have a particle with momentum k1 now, is there a way to change this particle's momentum, or if you apply again, you-- like in this picture, it's like you have another particle momentum k1.

PROFESSOR: No, there's no way to change the momentum of a particle. So once you created the particle, it just goes straight. Yeah, it just does not change anymore. Momentum for that particle is conserved.

STUDENT: Is there-- if you wanted to-- if you wanted to create a theory where you can change the momentum, is there a way, or is it just--

PROFESSOR: Yeah, there is a way. So after dealing with this free theory, and then we will consider the last simplest theory, and then that will introduce interactions.

STUDENT: I see. So--

PROFESSOR: And when you have interactions, then the particle momentum can change. Yes?

STUDENT: So the thing we're calling particles are like localized in momentum space, but not at all in position space.

PROFESSOR: Good. Yeah, these are the momentum-- yeah, it's like the plane wave in the non-relativistic-- yeah.

STUDENT: And if you were to try to localize it in position space like a Gaussian wave packet or something kind of like that, would you still be able to commute things and stuff?

PROFESSOR: Yeah, so the commute things don't change because everything is built by a k, and they always commute, and so that won't change. But indeed, we will talk about the wave packet to localize in space. Other questions? OK, good.

So also-- so the last-- we will talk a little bit about the technical point. It's that so far, we haven't talked about the normalization of such a state. So now, let's look at the normalization of state.

So let's just look at this state. The single particle state-- let's look at this normalization. So let's look at k with k prime.

So now, if you take the overlap of this with the k prime, and then you can reduce this to the commutator of a k and a k prime-- a k dagger and a k prime. So you will get just this. So again, this is a five-second calculation. So you find this if you do the overlap.

But you already did in your p set-- this thing is actually not Lorentz invariant. This is another good thing. This is another good object under Lorentz transformations.

But when we construct state, we would like to have our state to have good properties under Lorentz transformation. So we will choose a slightly different normalization. So instead of this state, so we will define the following states.

And we will define a k now without the vector to be the square root 2 omega k and this k. And so this is the 2 square root of k a k dagger acting on 0. And now, if you compute the overlap with this k and k prime-- so now, you have square root k-- square root omega k, for one of them, and then you have square root k for two of them, and then you have 2 omega k. And then you have 2 pi cubed delta 3k k minus [INAUDIBLE].

So now you recognize this object from your pset. The omega k multiply by this guy actually have good Lorentz transformation properties. So this guy actually transforms nicely under Lorentz transformations.

So this will be the normalization we will use from now on. So these states have very-- so indeed, you can show-- so that will be, I think, in your pset 3. You should look forward to it. So you can show if you act a Lorentz transformation-- so lambda is a Lorentz transformation, and u Lambda is the operator to generate that Lorentz transformation-- on such a state k, and then you just get lambda k.

And lambda k is the Lorentz transformation acting on that k. It's the Lorentz transform the k. But yeah, you should see your maybe p set 3, so-- if I remember it. Good. Any questions on this? Yes?

STUDENT: So I'm not sure in this case here why we assume that omega k is the same for both k and k prime?

PROFESSOR: Oh, because you have a delta function here. Yeah, because omega k only depend on k-- it only depends on the spatial part. Other questions? Yes?

STUDENT: Yeah, so the inner product of the k k prime is not Lorentz invariant. Is that just if you're listing one of the states?

PROFESSOR: No, just this guy-- when you-- it's not about whether it's Lorentz invariant. It's not Lorentz covariant. Just when you transform it, just this object-- when you transform it, it's very awkward.

Does not have very-- you can transform it. You can write down a transformation for this object. It just does not transform nicely. Then that means if this does not transform nicely, it means each of them don't transform nicely, and then just not convenient.

So this one have the property that when I act u on this thing, and then actually, you can just directly corresponding to the state with Lorentz transform with the momentum. Yeah, and then it's very easy. And this property does not apply to this one. It does not apply to this k.

OK, good. So you can also talk about a wave packet. So this is the plane wave. So this is like a plane wave. This is a momentum eigenstate. So we can also consider general-- say, single particle state will have the following form.

So we have the following form. We can write it as psi just as a superposition of this k. Again, you only need to integrate over spatial momentum, because only the spatial momentum are independent. And then some arbitrary function of k, and then on this k. So any single particle state, you should be able to write it this way.

So in particular, by choosing appropriate f k, you can construct a local wave packet. So you can-- so choose k-- you can choose f k to construct a localized wave packet in space. So again, we will have some exercises like this in your pset 3.

And also earlier, you could have objected when I called this to be a two-particle state. Because each is a plane wave. So essentially, they are not localized. They cover all space.

And in one sense, they corresponding to two particles. So now, you can solve this problem, say, by constructing using d k1 2 pi cube d k2. So by choosing appropriate function k1 k2, this k1 k2, you can construct two widely separated wave packets.

Again, it's in space, so you can really talk about-- so these are the genuine two particle states. And so that confirms actually this k should be a interpreted as the plane wave version of the two particle states. Good. Any questions? Yes?

STUDENT: So now that we don't have a position operator like we did before, what's the conjugate, I guess, operator to our momentum operator, or is there--

PROFESSOR: Good. That's a very good question. Indeed, there will be a problem in your pset 3 which asks you to show there's no eigenstate-- or there's no eigenstate -- there is no perfect. Yeah, so in non-relativistic quantum mechanics, you have this state.

You have this wave function, so you can localize at the point, but there's no analog of such kind of state in the relativistic case precisely because there's no position operator anymore. Yeah, so you will explore this a little bit more later yourself. Just with this formalism, you can explore all these questions yourself.

That's the key. Actually, you already have the power to do that. So now, let's say a little bit more on the structure of the Hilbert space. So, here the structure of the Hilbert space is a little bit special because there's no interaction. So you can just separate it into-- so the full Hilbert space, which I write as script H.

Then, you can separate it into the vacuum state, and then you have the one particle Hilbert space, and then you have the two particle Hilbert space, and they're all-- they don't have anything to do with each other because there's no interaction. So in principle, you can have infinite number of them.

You can have infinite number of them. So if you restrict to always a finite number of particles-- so the set of states-- so this is one particle, this is two particle, et cetera. So that's of states with a finite number of particles.

It's called the Fock space. In the Fock space, we only have state of finite number of particles. One can have arbitrary number of them-- as large as you want. 10 billion, 100 billion, but it will be finite. So finally, those k-- either this k or this k, they are just plane wave normalized because they're normalize by delta function.

So strictly speaking, they are not normalizable states. And so strictly speaking, they are not in the physical Hilbert space. So strictly, speaking k or k or any of those states of k-- or k1 k2 are not normalizable. They are only plane wave normalizable. So just like-- just as psi is equal to exponential ik x in non-relativistic quantum mechanics is not normalizable.

So they're convenient for various mathematical operations, but they don't correspond to genuine physical states. So not genuine physical states. So physical state, we have to consider for example this kind of state. And then you choose f k so that is normalizable.

So if we take this, and then-- so with psi, given by that, then the normalization of psi, you can calculate it, and then this is given by-- so you can easily guess the answer. So when you do the overlap exactly, you find that this given by the modulus of f k squared.

So if you do the calculation to calculate the norm of that state, you find that given by that. And then we can choose this to be finite, and then this will be then will be a normalized state. So for this reason, it's sometimes-- we don't do it very often-- can choose just a basis of f x, say, of f i.

So i runs some number. And then this alpha-- then you can define alpha i and then this will provide-- the alpha i will provide a basis of the single particle normalizable states-- a single particle state. So here is a fun fact related to the earlier question was asked.

So naively, here, you have-- if you look at the k-- so k form a basis. Let's just look at the single particle state. So naively, this k forms a continuum-- uncountable continuum of basis of states.

But once we impose this normalizable condition, means you need to choose the normalizable f. And then you can show the basis of this normalizable f actually is countable. So in the end, the Hilbert space-- the one-dimensional the one particle Hilbert space is actually generated by countable basis just like in your ordinary quantum mechanics. It's actually rather than uncountable basis like in k once you restrict normalizability. And so this is a fun mathematical fact.

Good. Questions on this? So the last few minutes, we talk about conserved charges. Quantum-- the role of conserved charges.

So classically, we say if an action is invariant under some symmetry-- some transformation-- some continuous transformation, say, of the infinitesimal form, so the alpha are the infinitesimal parameters. So the alpha label different transformations, and a label different fields. Alpha label different transformations.

And then you will get the conserved current J mu labeled by this alpha, so alpha label differently for each. So alpha label different symmetries for each symmetry, and then you have a conserved current, which is satisfy the J mu alpha equal to 0. The alpha equal to 1, 2, et cetera-- label different symmetries.

And then you can write down your Noether current. So we derive a general formula for the Noether current is given by the form, say, partial L-- partial partial mu phi a. And then f a alpha. So this is essentially the same as before.

Essentially, I just add the index alpha now to label. You can have more than one symmetries. And then minus k mu alpha. So k mu alpha is the derivative-- total derivative -- suppose -- delta L is given by epsilon alpha partial mu k mu alpha.

So the k mu alpha is the total derivative corresponding to that respect -- corresponding to that transformation. And then the q alpha, that will be the conserved charge for that current, so zeroth component. So this is the classical story. Now, I'm using a little bit more general notation for each alpha.

So now, the key thing-- so now the key thing is that at the quantum level-- so at the quantum level-- first, as we already seen in the case of the Hamiltonian and the momentum operator, so q alpha is time independent-- is a constant operator. So this is the fact that this is conserved. And also another important property of q is that when you act q alpha on your field, if you look at the commutator between this q alpha on your field, it actually generates finite transformation.

So turns out-- yeah, so if we have put a parameter here-- epsilon alpha. Yeah, so it doesn't matter for Q-- say, it's upstairs, downstairs. And then you get-- so you just generate the transformation for f-- for phi.

So this can be shown explicitly. So you can show this explicitly. We're running out of time now. So you can show explicitly just using this expression. So let me just quickly outline the way to do it.

So if you, say-- so let's look at the zeroth component of this J. So the zeroth component of the J just equal to partial L, partial, partial, 0 phi a and f a alpha. So let's consider the-- yeah, I think today we will not have time to finish anyway.

Yeah, let's just leave this to the next time. Yeah, let me just say that the quantum level, this will generate the infinitesimal transformation. And then you can also exponentiate this Q. You can introduce, say, exponential i lambda Q.

So if q is the conserved charge-- and then you can exponentiate Q like this. And then this operator then generates-- when it acts on phi, then generates finite transformations. So this takes you to phi a prime-- the finite transformations.

Anyway, so in your pset, for pset 2, you actually check yourself a couple of simple examples, and you will see this works. And in the next lecture, we will give you a simple derivation to show this actually works. It works in general.