[SQUEAKING]

[RUSTLING]

[CLICKING]

PROFESSOR: So, last time, we talked about path integral formulation of quantum mechanics. So let me just quickly remind you the main idea.

OK, so classically, if we specify the initial location-- say, if you add t prime, x prime, and then the final location, t and x so at time t, you add x, and then there's a unique trajectory, OK? And there's a unique trajectory.

So, quantum mechanically, you ask a different question. You ask, what is the transition amplitude if you say, at t prime, you add location x prime? And what's the transition amplitude at time t you add the location x?

And then the answer for this question, now, is equal to-- you just sum of all possible paths between t prime and x prime and t x, OK? So you just sum of all possible paths. And we sum weights, OK?

So, more explicitly-- and with the weight, you sum of all paths. And the weight is given by exponential i h S. So, for each path, you have an action. And then you just evaluate, for each path, the action.

And then, here, also, the explicit h bar here and because the action has the unit of h bar. Yeah, so this would be the quantum mechanical answer, OK? So this is a formulation of quantum mechanics using path integral.

And so we can write this more explicitly, mathematical notation for the summing over paths is the following. So we introduce a notation like this, which you integrate over all possible trajectories between x t prime equal to x and x t-- x prime and x t equal to x.

And then you-- with weight S x t. And the x t is a function of x t which you integrate from t prime to t. You have t double prime. So 1/2 m x dot squared minus V x, OK? So this is for the one-dimensional particle.

And, also, this-- so all the trajectories-- this fixed endpoint at t prime should be equal to x prime, and the time t should be equal to x. And, also, this integration is a shorthand notation for limit.

So this DX(t) should be understood as a limit, when you take angle to infinity, this factor, and divide it by 2 pi i delta t. So, essentially, you divide your paths by intervals of delta t. And then you integrate over all possible.

Yeah. OK, so you separate the paths from t, t prime to t by n segments. So this is the t i. And then the location will be x i. So this is t i, and location-- and then you integrate the x i over that, OK?

So this is the formulation of path integral for quantum mechanics. Any questions on this? Yes?

AUDIENCE: Can you do this computation in some arbitrary basis that's not necessarily a position basis?

PROFESSOR: Yeah. Yeah, in principle, you can also do this in momentum basis-- yeah, momentum space. And, also, you can actually also generalize-- yeah, we will talk about how you generalize this to other basis. But this is the fundamental formulation.

And when you write it in other basis, essentially, you reduce it to this basis because this is the most intuitive way to formulate it. Any other questions? OK, good.

So a simple example is just a free particle. When you have a free particle, then the S is particularly simple. You just-- integration of 1/2 m x squared, OK?

So we described that when S is equal to this-- and then this is like a Gaussian integral, OK? Because this is quadratic in x, so this is like a Gaussian integral. And so you can reduce-- so, in this case, you have a Gaussian integral. Then, schematically, you have something like this.

You can-- when x is of this form, you can write it in this general form. You can write the path integral in this general form. And the k, in this case, will be some delta function. Yeah, I wrote down k last time explicitly, and then this is like a Gaussian integral.

And in the Gaussian integral, we can just directly write down its answer under some constant divided by det of k. So this determinant is defined in the spatial functions and C is some constant. OK.

So you may also view this just as a lateral-- this also just follow from this continuum limit, OK? It follows from this discrete case, which you take a continuum limit. Yes?

AUDIENCE: So when you evaluate that Gaussian integral, there's an i in front in the exponent.

PROFESSOR: Right.

AUDIENCE: How does this converge?

PROFESSOR: Yeah, yeah. This is the same as you do the standard Gaussian integral with the i. Yeah, that integral, you can also do.

AUDIENCE: But shouldn't this diverge if it's just--

PROFESSOR: No, it's a-- yeah, just with a single-- so this integral, we can define, right? Yeah, this integral-- we know how to do this integral. It's the same as this one. Yeah.

So you have to do a little bit mathematical trick to do this integral. But this integral is defined. We can give a value. Other questions?

OK, good. So this is the path integral for quantum mechanics with one degree of freedom. So, now, we can just immediately generalize this to more degrees of freedom, OK? And there's nothing really change.

When you have more than one degree of freedom, you just integrate over-- say, if you have, say, three particles, then you just integrate the D X 1, dx2, D X 3 and then use the same action, OK? So it's just straightforward, generalize.

And then we can generalize to field theory because field theory, we just take the numbers of degrees of freedom to infinity, OK? So, we can now-- yeah. Let me just-- so to field theory--

So, in this case, we just need to replace in the different places by-- so, remember, so the dynamical variable for a single particle-- you have this operator. And then when you go to field theory, the counterpart of this is the phi, is your field variable.

And then-- so this quantity-- when you go to field theory, the corresponding quantity is phi x at some t and phi x-- say, phi prime x at time t prime. OK?

Again, the way to think about this is just you should view this x as label. And then this is just as if we have infinite number of positions, OK? You just have infinite number of phi's, which are eigenvalues of this operator at time t, at t prime, and, similarly, here, OK? So, again, this one-- you can write it in the form in terms of the position eigenstate-- so this is the position eigenstate in field theory.

OK, so this is the position of eigenstate in field theory. And then we can use the same technique we discussed last time. You just split this into many, many infinitesimal pieces by splitting the time interval from t prime to t and split into many pieces and then just do it over and over and insert a complete set of state of phi, OK?

And, now, you just-- so H is now, of course, is the field counterpart of it because H-- you could, too. But the details of this does not matter. Say, this is the Hamiltonian. You can just insert this into here, and then you just repeat the same procedure, OK?

Just keep in mind, now, you just have infinite number of degrees of freedom, OK? That's the only difference. Your rotation becomes a little bit more complicated. But if you keep a very clear mind, with the correspondence with the single particle case, then everything is the same, OK?

If you understand how to translate the rotation here to the rotation here, then everything is just exactly the same, OK? Just everything goes through. So I will not repeat that procedure.

And, now, you just find this quantity. Then, again, we can write this in terms of path integral. So, now, this can be written as in terms of path integral of phi.

Now, you integrate over all possible configurations of phi with is the boundary condition phi t prime x equal to phi prime x. And the phi t x-- equal to phi x. And, again, you just have exponential i S phi.

And S is the action for phi. So, now, the S-- is now-- so, in this case-- yeah, it's an integration of the Lagrangian density. And L is 1/2. Yeah, it's just what we wrote down before.

OK, some potential phi-- OK. So for any scalar field theory, these just work identically, OK? And, again, you just get the integration over the Lagrangian, which is become four integral of the Lagrangian density, OK? Any questions on this?

Good. So whenever you get confused about path integral in field theory, then try to translate into the language of a single particle by doing this kind of replacement. And then you will be able to settle your problem.

And then whenever you get confused about the path integral in this quantum mechanics, then just reduce it by a finite dimensional integral, and then you should be able to understand, OK? Just always reduce it to the simple case. And, often, your confusion can be understood in that simpler case, OK? Yeah.

Good. So, now, let me explain a little bit what this notation means, OK? So this said in words means you integrate over all possible configurations of phi between t and t prime. And the final configuration-- the initial configuration to be phi prime, and the final configuration to be phi-- phi x-- and the initial to be phi prime x, OK?

So let me just say, previously, DX(t)-- if you think about it in terms of discrete case corresponding to-- you just sum-- take the product of all the o at different time d x i, OK? So in the continuum limit, it's like you just take the product of all possible value of t and then integrate over the value of x at that particular value of t. So that's the meaning of this DX(t), which is up to a prefactor corresponding to that.

So, similarly, phi is the same thing. So, now, remember, phi x has now become the dynamical variable. So this now becomes sum over-- take product over t. Oh, yeah. No, no, no.

First, you think of take product over x. So this is a label of x. So x are the labels of phi, and you take all possible value of x. And then each of them is just like a-- OK?

So just like you have so many different variables, each x label one degree of freedom, OK? And then you have phi t. And, now, then you use this one. Then you have t x d phi.

Is it clear what this equation means? So this step tells you that you have many, many different degrees of freedom. So this step is just enumerate all possible degrees of freedom which are labeled by x.

It's just like your standard integral. If you have five-dimensional integral, and each-- and then you just have a product of five different variables, OK? Here, we have all different possible value of x variables. And that variable is a function of t. And then, now, we use this equation.

And each of them is like here, OK? And then, now, you have the product of all possible value of t. And then you integrate, then, all possible values of phi, say, at point x and t, OK? But, remember, t always is between t prime and t, OK? Yes?

AUDIENCE: If I'm understanding your notation for the first equation, the second equality-- what is plugging in--

PROFESSOR: You mean here?

AUDIENCE: The one above it. No, the one above it.

PROFESSOR: Yeah, yeah.

AUDIENCE: So that second quality-- so when you relabel i to the t, that's--

PROFESSOR: Yeah, yeah. Yeah, here, I just write it in the continuum form. So, here, I labeled it by t i. So each location t i, I have d x i in that form. But if I go to continuum limit and go to infinite limit, essentially, at each point t, I have integration of x.

And so that's roughly the continuum form of that. And, here, it's similar. OK? Good. So once you have learned this trick, do the reduction, and I think you will be able to settle all your confusions about these definitions, OK? OK.

So, other than that, other than this additional label x-- so other than additional label-- I should call it here label x. So the field theory path integral for a scalar field is essentially identical to that in quantum mechanics, OK?

So just to emphasize this point. So, here, actually, you can define this general. It doesn't matter what your V is, OK? You can choose whatever V you want. The story does not change.

Just like when we do the single particle case, it doesn't matter what this V is, OK? You can choose arbitrary V, if you want. But if we choose V to be-- when v is for the free theory, then-- in this case, of course, a particularly simple.

And then, in this case, the path integral is again Gaussian, OK? The path integral is, again, Gaussian. I will not write it again.

So, again, that formula-- and this formula applies just this case, in a more complicated space. K is the operator in a more complicated space, OK? So, now, you not only have the space of t, but you also have the space of x.

But once you generalize to some space of functions, it doesn't matter. This function becomes more complicated, OK? And so, conceptually, they're the same. Conceptually, they're the same.

Good? So, now, for interacting theory, then, essentially, then the quantum field theory then reduce to doing this path integral, OK? If you know how to do this path integral, and then, essentially, you know how to solve the theory, OK? You know how to solve the theory.

And so, as I said, in the free theory case, it's essentially reduced to a Gaussian integral. Then everything is simple, and we will go back to discuss that in a little bit more detail later. And, now, let's just think a little bit how to treat the interacting case.

So, recall, our goal is to compute this object-- the vacuum expectation value of time-ordered correlation functions, OK? We want to compute this object, OK? So, now, we will discuss how we use path integral to compute this object, OK?

So before doing that, let's first try to do this in-- so before doing that, let's first discuss how to do this in quantum mechanics, before doing this in field theory. And we can just first understand how we compute the similar object in quantum mechanics. And once we understand how to do that, again, the generalization to field theory will be straightforward. OK. Good?

So, first, we talk about time-ordered correlation function in quantum mechanics, OK? So we will be introducing a number of tricks, and those tricks will then take over to field theory, OK? They will be carried over to field theory.

So, again, let's just consider this system, OK? Let's consider a system like this-- just with the Lagrangian of this form, just one particle theory, OK? And then the analog of this object is this object-- let's call it G n-- would be, say-- so let's call the 0 is the vacuum of the field theory-- of this quantum mechanical system.

And then you have time-ordered, and then you have x t 1 because the dynamical variable here for quantum mechanics is just x and x t n at different time and then here, OK? So we want to compute such object in quantum mechanics. OK?

So our goal is to develop techniques-- yeah, so you should imagine all these are operators, OK? All these are operators. So our goal is to develop techniques to calculate this.

OK, so before doing that, let's first understand how to do this time-ordering using path integral, OK? So before I do that, do you have any questions? OK.

So it turns out, actually, path integral is some of the natural, the most natural objects to think about such kind of time-ordered correlation functions, OK? And we will see, in a minute, the path integral is actually the most natural thing to-- the most natural framework to think about this kind of time-ordered correlation functions.

So before doing that, let's consider one simple example. So, firstly, consider-- so let's consider some other time, t 1, between t prime and t, OK? So we consider again, we go back to this problem, OK? We go back to this problem. But we can see there's some time in between t prime and t, OK?

So, now, let's consider this object. So the simplest case of this one-- you just take n equal to 1, OK? You just have one of them. Let's consider the simplest case.

So let's consider this object. So this is easy to do. We can just use the same technique we used before. So, essentially, most of the tricks in quantum mechanics just reduce to one trick.

Yeah, I say most of the case, OK? Most of the case, tricks in quantum mechanics reduce to one trick. What is that trick, which we already used over and over, say, in driving the path integral?

AUDIENCE: Identities?

PROFESSOR: Yeah, insert identities. So if you know how to insert identities at the right location at the right time, and, essentially, you can-- yeah, you know all the tricks in quantum mechanics. And so, here, we do the same thing.

So since this, at t 1, so we just insert the complete set of position eigenstate at t 1. So let's just do this. So we integrate x 1, t 1 and the x 1, t 1, x hat, t 1, x prime, t prime, OK?

So these are the-- so this integrates to 1, OK? This integrates to 1. So since this is eigenstate at t1, so this act on this will-- you just get your eigenvalue.

So we just get d x1, x1. So you just get eigenvalue. And then you had xt, x1 t1, x1 t1, x prime, t prime, OK?

But, now, we know how to do both of them. We just plug in our path integral formula to both of them, OK? So you can do this explicitly-- plug in in this expression there, OK? Plug in this expression there.

But it's also very intuitive. You know what this looks like, OK? So this is like the following. So, previously, we just do all the paths between these two, OK?

So, now, we are-- now you have this x t, and now we have t1. So suppose this is the time, t1. And suppose t goes up, and so this is t prime. So suppose this is the time t1, and this is t at prime, x prime, OK?

So this corresponding to you-- you first do the path integral to some location t1-- integrate over all paths here. And then you multiply x1, and then you do all the path integral here, OK? And then you iterate over x1, OK? You integrate x1, OK?

So, now, if you-- without this x1-- if we are without x1, then, of course, this is trivially equal to the previous one because you do all possible paths to t1 and all possible paths from t1 to t, and then you iterate over all possible location here. Then it's the same as you integrate from here to there-- the arbitrary path, OK? So the only difference is, now, we multiply by x1 and the value at t1.

And, now, we can immediately write down, using path integral, what is this object? So this object is, essentially-- OK, it's just equal to-- so, now, I will use a simplified notation and say, here, write x prime, t prime, x t to be the limit, OK? And then, again, we integrate all possible paths between them. But, then, we just have x t1 here.

You integrate over all possible paths between them. Just add, this time, t1. You multiply in the integrand the value of x at t1. Yeah, so this is essentially just the x1 here, OK? Is this clear?

You can do this explicitly by plugging in those formulas and then manipulating. You will get this. But it's much easier to understand it heuristically using a diagram. Good?

So this is very, very suggestive. It tells you, when we insert an operator here, what we do is just we translate the eigenvalue of this operator, plug it in the integrand. So, now, you can do the same thing.

Suppose you have two operators, OK? So, now, let's look at-- you have two operators. OK?

So you can almost immediately write down-- if you try to generalize that, what would you write down this, the answer? Yes?

AUDIENCE: State the x and t1 and the x and t2 as an integral.

PROFESSOR: Good. So you just-- DX(t). But this is almost correct-- not completely correct-- for one reason, which, actually, one of you asked before. Yes?

AUDIENCE: Just because you haven't ordered them.

PROFESSOR: Exactly. This is only equal to that only for when t1 is greater than t2. So, remember, all these paths-- they don't come back in time, OK? The paths-- they go forward in time. They don't come back.

So, here, it's like you have to insert the two-- here, you just insert the two sets of complete states. One is here. One is here. But the path integral cannot come back, OK? So

The order here have to be the same as the ordering of the path integral, OK? It has to be the same ordering of the path integral. So that means this is equal only for t1 greater than t2, OK?

But the same thing happens but for t2 greater than t1. So suppose t2 is greater than t1. If you start this expression, then we can ask, what the corresponding operator form? So what do you think will be the operator form?

Yeah, so for t1 greater than t2, we have this equal to that, OK? But, now, let's ask-- suppose it's opposite-- t2 greater than t1. But, on the right-hand side, we still have this one. But what should be the left-hand side?

AUDIENCE: x t2 and x t1.

PROFESSOR: Exactly. We just exchange x2 and t1 because path integral always follows the time order. So that means the order here has always to be time-ordered. So that means that the correct formula, which applies for all t1, t2-- it's just you time-order them. You just time-order them.

So no matter for what the value of t1 and t2, if you time-order them so that the x which is the larger time always sit in the front, then it's always equal to this one, OK? So this, now, I'll just immediately generalize, OK?

So this tells you, because of the path integral-- in the path integral, the time only goes forward. The time only goes forward.

Then the time ordering naturally arises in path integrals, OK? So you can immediately generalize this.

So for any t-- for any t1, tn between t prime and t, then you always have x t time-ordered, x hat t1, x tn, 0.

And this is equal to x prime, t prime, x t, DX(t), and x t1, x tn, exponential i S x t, OK?

So that's why we say, earlier, that, using path integral, our goal is to compute this object. So that's why, actually, using path integral to go compute it is very natural because the time-ordering-- it's just very natural from the point of view of path integral. Yes?

AUDIENCE: Shouldn't the x be x' t'?

PROFESSOR: Oh, sorry, sorry. Yeah, yeah, yeah. Right. Good. Any questions on this? So this is a key formula. OK, so this is the key formula. OK? Good. Any questions on this?

So, now, we have got the time-ordering. OK, now, we want to compute. But here is the vacuum correlation function, OK? So we have to actually go to the vacuum. Here, it's in such between the position eigenstate, OK?

So, now, we have to see how to do this for the vacuum. So, now, let me just introduce a simplified rotation. So let me call this whole thing X-- capital X, OK? I call this whole thing capital X.

So, now, so we want to consider the vacuum correlation function. So the idea is very simple, OK?

Right. OK. So we want to consider a vacuum function. We want to interested in the G n for arbitrary t1, tn belongs to minus infinity to plus infinity, OK? So you want to compute this.

OK, and the G n, using our notation, we know equal to 0, X, 0, OK? So I denote 0 for that. So now, again, we insert a complete set of states, OK? Well, again, we insert a complete set of states.

We insert 1 equal to dx, x t, which is also equal to dx prime, x prime, t prime, x prime, t prime, OK? This t goes to plus infinity, and t prime goes to minus infinity into here, OK? So insert that into here. OK?

And then we find-- yeah, so let me-- I need the bigger space, so let me just write it down. So if you do that-- so this trick is general. It does not restrict to the vacuum.

So, in any case, evaluating any states, you can reduce it to the path integral by doing this trick of inserting these guys. So then we have Gn, now, just equal to-- now we have limit t goes to plus infinity, t prime goes to minus infinity. Then we have dx, dx prime.

So we insert one here and the one here, in both places. Here, we insert the upper line. Here, we insert the lower line, OK?

Then we have 0, x, t, then x t, x, x prime, t prime, and then x prime, t prime, then 0-- times. OK?

So, now, we just have-- yeah, so let me just save some-- so this then become dx, dx prime. So these two are simple. So, essentially, they just become the vacuum wave function, OK?

So we have psi 0 star x and psi 0 x prime. And then we have this path integral, which we already know how to do, OK? Then we have this path integral, OK? Then we have that path integral.

So you just insert-- so when we call star, you just insert the star here, OK? And, now, with t and t prime to infinity-- and, yeah, the reason we take t prime and t go to infinity is obvious because we want to include-- so, here, the t has to go between the t prime and the t.

So if we want to have arbitrary t-- and then we want the t prime to go to minus infinity and t go to plus infinity. OK? So we have used that-- the x prime, t prime 0 is just equal to psi 0 x prime, just the ground state wave function, OK?

So that means that-- so this trick actually works for other states, too, OK? You can do the similar trick. And you just essentially get two more integrals to do. And you take that path integral, and then you just integrate it over the initial and the final wave functions, OK?

OK, so the trick applies to any state-- can you see-- not just the vacuum correlation functions. Yes?

AUDIENCE: So does this still count as-- because this is still quantum mechanics what we're doing, your ground state-- I mean, this is an energy eigenket, and it's the ground state, or--

PROFESSOR: Yeah. Sorry. Say it again.

AUDIENCE: So since this is still quantum mechanics, I'm saying, this 0 here is like an energy eigenket, right?

PROFESSOR: Yeah, yeah. Yeah, it's a ground state. It's the it's the lowest-energy eigenstate, yeah. It doesn't matter. You can also take other states, too. It just will be some wave function there. It will be some wave function there. Yeah. Yes?

AUDIENCE: So [INAUDIBLE] someone else said, which evolves in time, then moves [INAUDIBLE].

PROFESSOR: Sorry? Oh, yeah. Yeah, if you not take-- yeah, that's a very good question. So if you don't take-- and so if you take other states, and then, in general, there may be some dependence on t prime and t in this wave function. Yeah, indeed, yeah. So you have to take the limit, also, in the wave function.

Yeah. But, here, we are using the simplification that the ground state is actually time-independent. Good. Yeah, that's a good question. Other questions? OK, good.

So this works, essentially, for any state. But for the ground state, actually, there's another trick which can simplify the problem, which you don't even have to do this two additional integral, OK?

You can actually directly-- so, for the ground state, for the vacuum, there's actually another trick to get rid of-- to make the-- these two additional integrals unnecessary, OK?

So, now, let me tell you how you do this trick. And this trick is very important, also, in quantum field theory because, if you do a harmonic oscillator, of course, you know the wave function. But if I give you a aharmonic oscillator, which we don't know how to solve the ground state wave function, and then this will be a nightmare because then you don't know the wave function.

And for quantum field theory, in particular, in the interacting case, we don't know the wave function, OK? So even though this formally gives you the answer, but, in practice, it's actually often not convenient to use, OK? But, fortunately, for the ground state, there is additional trick which, actually, you don't need to use the ground state wave function at all, OK?

So, now, I tell you this additional trick. But this trick is only specific to the vacuum, OK? You cannot apply it to other states, OK? So this is specific to vacuum.

So, now, let's forget about this thing, OK? Now, let's forget about this thing. Start coming back from here, OK? So, now, let's coming back from here. Now, let's coming back from here.

Just consider-- again, I take limit t goes to plus infinity, and t prime goes to minus infinity. Let's look at this object x t, x, x prime, t prime, OK? OK?

So, now, again, we are going to insert identity, OK? Now, we are, again, going to insert the identity. And, now, we insert the identity in a different way. Again, we insert the identity here and insert the identity here.

But, now, we insert identity expressed in the complete set of energy eigenstates, OK? So, now, we insert the complete set of energy eigenstates. I just formally label it by m. OK? So, now, let's just insert in the two places, OK? Yeah.

OK, so, now, we have two identities. Then we have n and m. We have n and m. So, now, I will-- so you keep this in mind. I will not copy this over and over, OK? So keep this limit in mind-- always, t prime. It goes to minus infinity.

So, now, we have-- and then we have x t with m, and m x with n, and n with x prime, t prime, OK?

So, now, if we look at this sum-- if n and m equal to 0, this is the object we want, OK? This is the object we want. But this also contains many, many other stuff.

So, now, we will use the trick to isolate n equal to 0 and m equal to 0 piece, OK? You isolate that piece. That's what we are going to do now.

So this is the commonly-used technique in quantum field theory-- not only in quantum field theory, actually-- in many areas of physics. So, now, let's first-- to explain the trick, let's look at this object. So let's look at this object, OK?

So this object is limit t prime goes to minus infinity, some energy eigenstate n and x prime, t prime. So, again, this is written in the Heisenberg picture. So we need to-- so translate to your more familiar language of the Schrodinger picture.

So we need to write it as n, and then you have expansion i H t prime and then x prime, OK? So, now, it is the standard Schrodinger picture state.

So, now, since n is the energy eigenstate, we can just act this one, OK? So this is just equal to limit t prime goes to minus infinity exponential i En t prime and x prime, OK?

So, now, we will try to select the ground state. And we do that by doing the following. So, now, imagine giving En a small imaginary part, OK?

OK, imagine-- yeah, take En goes to En 1 minus epsilon. So epsilon is a small positive number, which is equivalent to taking-- or, you just take H, your Hamiltonian, take into 1 minus epsilon, OK? Give your Hamiltonian a slightly imaginary part, OK? It's equivalent, OK?

So, now, with this-- and now we have-- so limit t prime goes to minus infinity. So, now, we have exponential i En, 1 minus i epsilon t prime, and n x prime, OK? So this is just equal to that.

So, now, you see-- suppose we normalize our state so that-- yeah, so, by definition, En are greater than E0 for n greater than 0, OK?

So the effect of putting epsilon here, when you multiply it, it still give you a factor like this-- epsilon En times t prime, OK? So give you a real factor like this. OK?

So, now, t prime goes to minus infinity. And the En is greater than E0. So for any state which is not ground state, this factor will be exponentially small-- goes to 0.

So the ratio between this factor and the ground state goes to 0 for any excited states, OK? Because of the En minus E0 greater than 0. And this goes to minus infinity, and the epsilon is positive, OK?

So this implies-- so we continue. Sorry. It's just-- so let's continue over here. So this means we get, essentially, 0 for n not equal to 0 and the exponential i E0 1 minus epsilon t prime for n equal to 0, OK? Yes?

AUDIENCE: If you want to formally show this-- that all energy levels that are not in a ground state get exponentially suppressed, do you divide by the ground state energy into the [INAUDIBLE] energy, or how do you do it?

PROFESSOR: Oh, you just take out the overall factor corresponding to the ground state. Yeah, then all the other coefficients wil go to 0. Yes?

AUDIENCE: What is the justification or why are we allowed to add a small imaginary part to the energy?

PROFESSOR: It's a mathematical trick.

AUDIENCE: OK.

PROFESSOR: At the end of the day, we set epsilon go to 0. And then you go back to your original Hamiltonian. So this is just a trick. If you're doing this order properly, it just naturally selects the ground state for you. Yeah. Yeah, just a pure, mathematical trick. Yes?

AUDIENCE: Epsilon has to have a certain sign, right?

PROFESSOR: Yeah, epsilon-- when we write this epsilon, we always assume epsilon is greater than 0. So, yeah. You always assume-- yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Hmm?

AUDIENCE: I don't see why you can do that [INAUDIBLE].

PROFESSOR: Oh, I can-- I have a path integral. And, essentially, I can design the rule I want, as far as, at the end of the day, I get the desired quantity. Yeah. But the order of limit is very important. You have to be very careful. Yeah. Yes?

AUDIENCE: Can you get around this with some sort of a stationary phase or saddle point expansion? Because, in the end, you're going to be integrating exponential factors. And so the term that dominates is--

PROFESSOR: No, here, it's not-- no, here, you cannot do a stationary phase because if you just-- yeah, yeah. You really need to suppress the other contribution-- not only suppress them. Just, they have to go to 0. Yeah-- genuinely go to 0.

OK, so, similarly, with the same trick, just the same thing, in one shot, it selects n equal to 0. But you see, actually, it also selects m equal to 0 because, you see, with the same sign, you can easily check yourself when t goes to plus infinity.

So these now become limit t go to plus infinity, say, exponential minus i En, 1 minus epsilon t times x n. And then, again, this become 0 for m not equal to 0.

And then the ground state-- for n equal to-- for m equal to 0, OK? So this simple trick beautifully selects the m equal to 0 and n equal to 0 piece, OK?

So, now, we can write-- so this is just the ground state wave function. And this is just the ground state wave function, OK? So, now, we have limit t prime goes to minus infinity, and t goes to plus infinity, x t, x, x prime, t prime.

So, now, you go to-- so you have the ground state wave function star, x prime, x, now 0, x, 0. And then you have this phase factor. And with everything else, goes to 0, OK?

So we get that, OK? And so this is the-- so, now, after those factors, we get what we want, OK? We get what we want.

So, but, still, you have lots of lowering factors. But they can be easily got off. They can be easily get rid of using the following.

So, now, you also-- similarly, the exact same reason, whether-- so in this discussion, whether you have x there, or what's the form of x? Doesn't matter. x can be anything. So x can be anything.

So you can just take x equal to 1, OK? And then you also have limit t goes to plus infinity. t prime goes to minus infinity. The x t, x prime, t prime-- so this is also equal to psi 0 x, psi 0 star, x prime.

And, now, it's the 0 with 0 because the you've set x equal to 1, OK? And then you just ground state. And so this is just equal to 1. And then you have the same factor, OK?

So, now, we can just find G n, OK? So this object is Gn. So Gn, then, can be defined as the ratio between the two.

We take the limit-- t plus infinity. t prime goes to minus infinity. Have x t, x prime, t prime, and x t, and x, x t, OK? You can just find that as a ratio between the two.

So this gives you the expression for calculating this correlation function. And the x is any time-ordered product, OK? And so, now, for each of them, we know the path integral. And then you can just write down the path integral for each.

So, now, you see this G n is, in principle, a physical observable that's actually involving the ratio of two path integrals-- one without any x and one with some x, with your desired correlation functions. So I mentioned before that when you evaluate the path integrals, you will get that kind C determinant or those things. They just cancel between upstairs and downstairs. Later, we will see explicitly, OK?

Yeah, so this is a beautiful formula, OK? Yeah. And then you don't have to-- yeah. Yeah, so all those factors canceled. You don't need to know the ground state wave function. You can just perform that path integral.

And here, in principle, you can also take arbitrary x and x prime, OK? You can take arbitrary x and x prime. It doesn't matter.

So, conventionally, we just take x equal to x prime equal to 0, just for simplicity, OK? We can just take them to 0. Good. Any questions on this? Yes.

AUDIENCE: Why can you specify both points in your [INAUDIBLE]?

PROFESSOR: Yeah, I just-- the only thing matters is there-- so they only come into this prefactor, and then cancelled between the two. It doesn't matter. So all the factors there and these factors, they just cancel. And you're just left with this x0 left, so it doesn't matter. They cancel anyway, so it doesn't matter where you put x and x prime. Yes?

AUDIENCE: So we didn't actually take the limit as epsilon moves to 0, or anything, so it doesn't actually matter.

PROFESSOR: So, normally, when you calculate that at the end, and then you take epsilon goes to 0. So we just, normally, leave that implicit. Yeah, when you do the path integral, you keep a small epsilon. So after you finish, in the end, you take the epsilon goes to 0. Yes.

AUDIENCE: Sorry. Also, on why would you take x and x prime equals 0-- doesn't our path integral-- the bounds-- depend on x and x prime that you choose?

PROFESSOR: No, the bound does not depend on it. The path integral is only limited by t prime and t. Yeah, initial value, they are the same-- of course, specific value of this path integral will depend on x and x prime. But they cancel between upstairs and downstairs. So they cancelled. Yeah.

And so, in the end, this right-hand side does not depend on x and x prime. Yeah, also, you see here, if this formula makes any sense, the left-hand side does not depend on x and x prime. So the right-hand side must not depend on x and x prime, otherwise, this formula immediately wrong. Yes?

AUDIENCE: [INAUDIBLE] have to keep the small epsilon back in [INAUDIBLE]. So, again, we take limit as epsilon goes to 0.

PROFESSOR: Yeah, that's right. That's right. Yeah, just when you do the calculation, you keep a small epsilon. When you do the path integral, you keep a small epsilon. But when you finally calculate the final answer, you can set epsilon to 0.

AUDIENCE: Do we do that before we take the t to infinity, also, or does it matter?

PROFESSOR: No, you have to keep an epsilon. This, you have to take first. So this is part of your calculation. You have to always take this first because the final answer also does not depend on t and t prime. And because this is the vacuum, this correlation function does not depend on your t and t prime.

So you always set the-- yeah, you keep a small epsilon, take this limit first, and do the calculation. Do the calculation. And then when you calculate everything, at the end of the day, then you set epsilon to 0. And then that guarantees you chose the vacuum.

Good. There's one more trick we have to do, OK? Sorry. There are too many tricks. There's one more trick we have to do.

So, now, we have told you how to calculate this guy for arbitrary x. But, often, it's actually not-- it's not the smartest way to directly calculate such an n-point function, OK? It's often not the smartest way.

So there's something a little bit more clever. It's called generating functional. So some of you may have seen this, say, in your high school math competition.

So suppose we want to do an integral like this, OK? So let's do-- again, let's go back to our ordinary integral. So let's just suppose we want to do integral like this. So do you have a parallel? Let's just say-- say this. We have some phase, like what you do in the path integral.

And we want to calculate the x to the power m OK? n is some integer. Yeah, here, like you have an n-point function-- suppose we want to calculate this.

So you can just calculate this guy. It's fine, OK? But there's a better way. Often, there's a better way. Instead of calculating this for individual n, you can actually consider the following quantity, Z a, which is defined to be dx exponential i lambda fx, then plus i-- say ax, OK?

So you consider this object. so this object, when you expand this exponential-- yeah, so maybe I will-- I think I don't need this anymore. Yeah, let me just erase this.

So, now, if you-- so why we are interested in this object? Because imagine you expand this exponential factor, and then, essentially, you get Z a is exponential n equal to 0 to infinity, n factorial i a n, Zn, So?

If you expand this thing in power series of a, a power series-- if you expand this and then just n-th term will give you an x power n. And then it's the same in that we just expand this Z a in power series, and then the Z n would be the n-th coefficient, OK? So if you compute Z a in one shot, and then the easy Z n, you can just do by a Taylor series. OK.

So, sometimes, we also write Zn as, say, i to the power n. Then you take n-th derivative of Za over a, and then you set a equal to 0, OK? So this is the same thing.

So when you take n derivatives of a, you get rid of the earlier-- the lower powers. And then the higher powers you get rid of by setting a equal to 0, OK? And then let's just pick this term, pick Zn term.

So Za is called the generating function. OK? Generating function. Oh, OK. So, now, when we want to-- so one final remark.

So, now, we want to compute this guy. So this Gn is equal to t, x1 to xn, so x t1 and x tn. Again, instead of computing each one of them, we can consider such a so-called generating function or generation of this idea.

We can consider Z J t. We can consider this object-- DX(t) exponential i S, x t, and, now, add i dt from minus infinity to plus infinity, J t, x t, OK?

So, remember, this is just the path integral. So let's just look at the upstairs, OK? Exponential x, exponential i S x(t).

So, now, this starts the same thing. So, now, if you take this, imagine you expand this factor and then to n-th power. Then you will generate this term as a coefficient, OK? We will discuss that in more detail next time, OK?

So, often, instead of computing this object directly, we compute a generating functional, so-called generating functional, which often makes things much easier, OK? Yeah, so with this preparation, and then we are ready to tackle how to treat this thing-- how to actually find such object in quantum mechanics.

And once we have that, doing field theory will be automatic. And then you will, then, know how to actually calculate the scattering amplitude, which is a good achievement. OK, so let's stop here.