[SQUEAKING]

[RUSTLING]

[CLICKING]

HONG LIU: OK. Let us start. So last time, we started talking about discrete symmetries. So if you have a complex scalar theory-- so if you have a complex scalar theory and then, in addition to U1 symmetry, classically there are also discrete symmetries-- firstly is parity.

So we normally denote with a symbol P and the act on x mu by t taken to minus x. OK? So you just flip all the spatial direction. And you also have time reversal. OK?

Time reversal. And then I normally write as a script T on x mu, then take to the minus t and x. OK? It does not do anything with x. And then we also have a charge conjugation, which takes phi and goes to phi star. OK?

And so of course, our goal is to understand the symmetries at the quantum level. OK? So at the quantum level-- so you're already-- then in quantum mechanics, symmetries are-- symmetry transformations. So they are corresponding to unitary, OK, transformations. OK?

OK? They corresponding to unitary transformation. We already have done that for continuous symmetries. And the same thing is with the discrete symmetries. And in quantum mechanics, the implementation of the discrete symmetries are also through some unitary operator.

And so there's a slight subtly with the time reversal because all the other symmetry transformations corresponding to the unitary transformations, except for the time reversal. That actually is corresponding to the anti-unitary transformations. So have you seen that in quantum mechanics? OK. OK. Let me just review it here.

So yeah. So previously, I want to say "recall." But we don't have to recall it. So say-- so let's denote the UT as the unitary operator. So this is the unitary operator for time reversal. OK?

And then the claim is that this has to be-- this should be anti-linear. And it's not actually ordinary unitary. It's called the anti-unitary. OK? And so now let's try to explain where those things come from.

So suppose that you have a system which we say is time reversal invariant. So by definition, the symmetry is that the corresponding transformation of the symmetry should commute with your Hamiltonian. OK?

So this is what we mean by the system to be time reversal invariant. OK? So here, I'm just talking about it just in quantum mechanics. OK? Let's not even worry about the field theory.

So now-- so suppose we have a time reversal invariant system, which the time reversal operator commutes with your Hamiltonian. And then let's act this UT on the wave function. OK? Then you get some psi prime.

So by time reversal, we expect physically, OK, if psi satisfy the standard Schrodinger equation, and then psi prime should satisfy the Schrodinger equation with t goes to minus t. OK? It's the same Schrodinger equation, but with t goes to minus t. OK? So that's what we mean by time reversal. OK?

So now let's try to do that. So let's start with this equation. And now let's just act by UH from the left on this equation. OK? So the left-hand side has UH.

Yeah. Yeah, it means that-- yeah, yeah. Before too [? I leave it ?] do one more step-- which means that the psi prime should satisfy the following equation. OK? So you just take t equal to minus t. And then you get the additional minus sign. OK? So psi prime should satisfy this equation.

So now let's see how you get this equation from here by using this relation, OK, by using this relation. So this is simple. Let's just add a U on the left of this equation. So let me call this equation star and this one star-star.

So if I add UT on star, then on the right-hand side, UT commute with H. OK? So on the right-hand side, we just directly get UT H psi.

And we can just commute them. So this is the same as H psi UT-- UT psi and this is just psi, psi prime. So the right-hand side just gets psi prime. OK? And then from the side, you get UT i partial t psi. OK?

So in the ordinary situation, we would have trouble. So if you-- because this is the operator. i is a c-number. And the partial t is some number. So you can just directly pass through this i and the partial t.

But when UT act on psi, then you get the same equation rather than the equation with a minus sign. OK? Then you have trouble. And the only way to get that equation is to require-- we require-- when UT pass through i, take it to minus i. OK? We impose this condition. When UT pass through i, you get a minus i. OK?

So this means-- so we can define anti-linear operator. Anti-linear operator is defined to be the following. It means that if you have some operator a act on some number c, when you pass through that number c, you get c star a. OK? So anti-linear operator a is defined to have this property. And c is just an ordinary c number, OK, an ordinary c number.

So you see here UT. Is order for UT to be ready to be a time reversal operator. And UT had to be anti-linear, OK, so that, when you pass through i, it takes i equal to minus i. Because the minus i is the complex conjugate of i. OK? Good. Any questions on this? Good?

So yeah. So UT should be anti-linear. OK. So also, for anti-linear operator, the adjoints should be defined differently. Yes?

AUDIENCE: [INAUDIBLE] you started off with trying to show star-star using this-- how UT acts on star. But then you kind of inserted this requirement ad hoc to satisfy star star.

HONG LIU: That's right. Yeah, that's right.

AUDIENCE: So I'm a bit confused. Then how do you come up-- why the star-star [INAUDIBLE]?

HONG LIU: No, the star-star is the definition of time reversal. Right? Star-star is what physically we mean by time reversal. If you want something to implement the time reversal, it has to achieve star-star. Yeah. Other questions? OK. Good.

And then we derived that in order to achieve the time reversal, then UT has to be anti-linear. And then because ant-linear operator has this weird property, the adjoint of anti-linear operator-- the definition should also be changed. OK?

So the adjoint of anti-linear operator A is defined to be, say, if you have psi, A dagger chi. OK? If A is an ordinary operator, and then the definition of the adjoint is that this is equal to A psi chi. OK? So this is a standard definition. Yes?

AUDIENCE: Yeah. So I was just confused about how you deal with making multiplication, scalar multiplication noncommutative and I was wondering why you couldn't do-- wouldn't it make sense to just say that UT and H anti commute?

HONG LIU: UT and H anti-commute? So by taking the-- yeah, you can also do that. It just does not work as well. Yeah. You see whatever it works. Yeah, yeah. It's much easier just to make the UT to be anti-linear. Yeah. Yeah. Yeah, yeah.

For this particular equation, it can work. But then you have to be against the principle that somehow H commute with the symmetry. Yeah, yeah. That is considered to be more sacred a principle. Yeah. Other questions? And indeed-- yeah, yeah. Right. OK. Good.

So the way we define the adjoint should also be different. So this is standard definition. So for anti-linear, you have this definition. And then you put the additional star. OK? You put an additional star. And the reason you put the additional star is precisely due to this property. OK?

Because remember, if you put a complex conjugate, put the c number here, and then you should be able to take the c number outside this kind of overlap. But in order to be compatible with the property of the anti-linear operator, then you have to define the adjoint to be this way so that you have a consistent story. OK? Just I will leave it to you yourself as an exercise to show this is actually the right definition. OK?

And then we can define the anti-unitary as that, if you have an operator with anti-unitary, it means U phi, U chi. So standard unitary condition is just that this is equal to chi-- and the psi and the chi. OK?

And now for the anti-unitary, again, you just put the star. OK? You just put the star. OK? So the claim is that the UT, the time reversal operator, should be an anti-unitary operator, uh, in order to be a symmetry, in order to be a symmetry. OK?

So this is just a brief review of time reversal in quantum mechanics. So now let's look at the quantum version of those symmetries in quantum field theory. OK? So first, let's do the warm-up. Before let's do it for the Dirac theory, let's warm up for complex scalar fields, which is much simpler. OK?

So in this theory, the action is invariant, we said, under those symmetries. OK? If we just flip the spatial direction, clearly that Lagrangian is invariant. If we leave the time direction because the time derivative is quadratic, it's also invariant. And again, if you change phi to phi star, the Lagrangian is invariant. OK.

So quantum mechanically, those symmetries should be implemented by some unitary operator. OK. So now the parity transformation on the field to the corresponding to you start with phi x. And then you go to phi prime x. So phi prime x should be equal to phi px. OK? This means that the-- and then just put-- yeah, you can also put the phase here. OK?

So you can show that this transformation certainly is a symmetry of this Lagrangian. OK? Yeah, because as I said, if you just flip it, it does not change anything. But you can also put an arbitrary phase here in principle. OK? You can in principle put an arbitrary phase here. Because they all only depend on the phi and phi dagger. OK?

And so this is p. And then this is the t. Then corresponding to phi x, then it goes to some phi prime x equal to, again, some phase-- so phi tx. OK? And then the charge conjugation normally is called C.

So parity is normally called the P. OK? So here, it's corresponding to phi x, goes to phi prime x equal to eta P phi star x. OK?

So in each case, such kind of transformation should be generated by some unitary operator. So for example, this should be generated by UP phi x UP dagger. OK? And then similarity for this one, we call the UT-- UT dagger. And here, you see it in UC dagger similarly. OK? So at the quantum level, this should be generated by some action of some unitary operator. Yes?

AUDIENCE: Do you always have this U1 symmetry or do you have -- is there an interacting theory where you have one power of phi. But we don't have--

HONG LIU: Sorry?

AUDIENCE: So you can add an arbitrary phase?

HONG LIU: No, no. I'm going to talk about the phase. OK? I'm going to talk about the phase. So if you are even looking at that Lagrangian, and then you see you put the phase, this is invariant. OK. But now the question is, can those phases be arbitrary? OK? The question is then, can those phases be arbitrary?

So in the case of this complex scalar, actually it should not be. So let's use the example of the parity. So we know that when you do parity twice, OK-- if you do parity twice, you should go back to your original. Yeah, your system does not change. OK? If you just do reflections while your system does not change.

So that means that, when you do this twice, you should-- you haven't changed anything at all. OK? So when we do it twice, the phi should come back to itself. And indeed, this P-squared, if you do it twice, comes back to itself. But then this phase should also come back to itself. OK? So that means that the-- eta P-squared should be equal to 1. OK?

So if you wanted the symmetry to come back to itself and then eta a P, then it can only be plus-minus 1. But it actually can be minus 1. OK? It does not have to be plus 1.

So recall when eta P is equal to plus 1, we call it a scalar. When eta P is a minus 1, we call it a pseudoscalar. We call it pseudoscalar. OK? And a similar thing can be said about the eta t and eta c. OK?

So by definition, this unitary operator acts on phi. It should act as this. So now we can-- and now remember the phi is expressed in terms of the creation and annihilation operators. OK?

So by requiring-- yes?

AUDIENCE: [INAUDIBLE] scalar or pseudoscalar is it true that if eta P is equal to one then eta T and eta C also be 1?

HONG LIU: No, no. They don't have to be. No, no. They don't have to be. They can be-- say this equal to 1. eta T can be minus 1. Yeah, yeah. They don't have to be. Yeah.

AUDIENCE: [INAUDIBLE]

HONG LIU: Yes?

AUDIENCE: If eta is one, its a scalar in what sense, I guess?

HONG LIU: Huh?

AUDIENCE: It's scalar in what sense? Especially for the other eta's don't have a [INAUDIBLE] Like, I don't understand what is a scalar it is equal to 1.

HONG LIU: You just say it's a convention. It's a name. Yeah, we just want to distinguish these two cases. So this is-- so when you say it's a scalar, we should say this is the ordinary scalar.

s this is called a pseudoscalar. Yeah. We just want to distinguish these two cases. Yeah. Other questions? Yes?

AUDIENCE: So if [INAUDIBLE]

HONG LIU: Yeah.

AUDIENCE: [INAUDIBLE]

HONG LIU: Yeah. So it's getting complicated. You can you can try to invent eight names for these, where there are eight possibilities. And people normally don't bother to invent eight names for it.

And since the parity is often used, and so they just talk about the parity. OK? They just define it this way. We normally don't give a name, let's say, in this case. Yeah. You just specify whether it's 1 or minus 1.

And so the reason we give name for this is because of the pion. And pions are pseudoscalars. OK? They actually transform under the parity-- by minus sign. The Higgs would be the ordinary scalar that transforms as 1. Good?

So if I have this expression-- so by requiring UP phi dagger phi x UP dagger equal to that thing-- yeah. You could do this, say, plus-minus eta P phi Px. So by requiring this, we can work out how UP act on a and b. OK?

We can work out how UP worked on a and b. So you find that the parity acting on ak UP dagger equal to the same eta P factor. And then take it to a minus k. OK?

So this makes complete sense because we integrate the k as momentum, right? So under reflection of a spatial direction your momentum also change direction. And so it takes ak to get to a minus k. And then, similarly, you can work it out the UT acting on ak UT dagger. It gives you eta T, also a minus k. OK?

And similarly, with the bk. OK? So this also makes sense. When you reverse your time direction, your momentum also changes sign. OK? Your momentum also changes sign.

So now the charge conjugation is a little bit more interesting. OK? So the UC for the charge conjugation is a little bit more interesting. So the charge conjugation takes ak dagger equal to eta C equal to b. OK? And similarly, it takes the-- yeah.

So you see that what charge conjugation does is to exchange particle and antiparticle, OK, exchange particle and antiparticle. Good. Any questions on this scalar field? Good.

So now let's look at the-- with this scalar field story, now we can look at a Dirac field, which is more intricate, OK, which is more intricate. So let's write down the-- so let's forget about the previous thing. Let's write down the action for the Dirac theory. OK?

And then let's also write down this equation of motion. OK. Now let me call this equation star and erase the earlier star.

So now let's first understand how the parity should act on the Dirac field. OK? So now, actually, everything with the Dirac field story becomes tricky. OK? It becomes tricky. So naively, if you wanted to just try to generalize the scalar story, we say, let's imagine the Dirac field should transform. OK?

I'm just-- to say for the parity, for example, psi Px, OK, with some phase, OK, maybe with some phase. So we want to naively generalize to the scalar story, that would be what you do. OK? That should be how the parity transform.

But now remember psi now is a four vector. Nobody tells you that, under parity, the different component of this four vector cannot get reshuffled. OK? Remember, under Lorentz transformations, they do get reshuffled. OK? In Lorentz transformations, remember that for spinor the Lorentz transformation not only act on x but also on the internal space of the spinners.

So you would expect the most general parity conservation should also act on the internal space. OK? So we would expect the phase is not enough. So we actually have to put a matrix here and that's called D. And whatever phase will be absorbed in this D. OK? Of course, you'll also include the phase.

And yeah. So for notation convenience, I will also write it as psi prime x prime equal to D psi x. OK? So the x prime equal to Px. OK? So these two are the same equation. OK? So you just write, D is a matrix acting on spinor space. OK?

So in what sense we say the Dirac theory is parity invariant? OK? Let me just formulate it in terms of the equation of motion. You can also equivalently reformulate it in terms of the action. OK? Yes?

AUDIENCE: [INAUDIBLE]

HONG LIU: Oh, we will talk about the behavior. This is a good question. It does not have to. And we will talk about the property of T. Yeah. Do you have other questions? Good.

So what do we mean by the theory is parity invariant? OK? By the theory-- we say Dirac theory is parity invariant. We say if Dirac theory have a parity symmetry, you can always define a parity operation.

So when we say Dirac equation have a parity symmetry, it means that if we start with a psi which satisfies the Dirac equation, and then there should exist such a D, such a matrix, such a transformation so that at this transformed psi still satisfies the Dirac equation but in the reflected space. OK?

So you want to reflect. So x prime goes right into the reflected space. And as we say the theory have a parity symmetry-- it means in the reflected space you should have the same equation. OK?

You should have the same form of the equation. So that means that this psi prime, when you're writing it in the reflected space, should have the same form of the equation as the original Dirac equation. So this is the same concept as this Lorentz covariance we described earlier. OK? Yes?

AUDIENCE: Why doesn't the parity transform the gamma matrices matrices? Because we changed d mu but not the gamma?

HONG LIU: Yeah, yeah. Gamma matrix is just some numbers. Right? Because the gamma matrix is always just some numbers. They're not spacetime variables. And so they're not-- so they don't change under the parity. Other questions? Yes?

AUDIENCE: Is there any reason why D has to be a matrix? Why does it have to be linear?

HONG LIU: No, no. D does not have to be a matrix. It's just that D can be a matrix. But we want to find the D so that we have this. But D can be identity.

AUDIENCE: Could it be some complicated function, like non-linear? Because we already allowed for non-linear, like, time reversal operators [INAUDIBLE]

HONG LIU: No, no. The time reversal is linear, right? Time reversal is linear.

AUDIENCE: Antilinear.

HONG LIU: Yeah. The action of a-- normally, action of a-- yeah, this kind of-- just go to psi itself. But you can-- yeah. You're asking why not-- why we don't use a psi squared term? Or you are asking whether the D can depend on spacetime? Which one are you asking?

AUDIENCE: Yeah. I mean, psi-- like, some arbitrary function of psi.

HONG LIU: Yeah. Yeah, but this is the simplest one. Yeah. Yeah, if we can make this work, then you don't have to do that. Yeah. You do the simplest one first.

Also, we do believe that this kind of parity operation should be a linear operation. Yeah. Yeah, if you add the two operators together, you do the parity, and you should-- after you do the parity should be still summed together.

If it's a non-linear function, then you won't satisfy that property. Yeah. Yeah, just from our experience, the-- yeah, just physically, we expect the parity to be a linear operation. Good? OK.

So we want to find the D so that this is satisfied. OK? So that means that the-- so this is easy. So now let's just look at this equation a bit. OK? So let's write it more explicitly.

So partial mu prime-- yeah. So we write this explicitly. So partial 0 prime is the same as the original partial prime because you don't flip the time direction. OK? And then you flip the spatial direction. So that means this becomes a partial i, partial i -- gamma i partial i. And so I remove the prime, but I have a minus sign here, OK, because you've changed signs.

And the minus n-- and this one from our definition should be equal to this. So this is just equal to D psi x equal to 0. OK? OK?

So now if you compare with this equation-- so the-- so now if you compare with this equation, now let's add the D on this equation. So that corresponding to D-- gamma 0 partial 0 plus gamma i partial i minus m psi x equal to 0. OK?

So this is the original equation. And now we want these two equations to be equivalent. OK? Because given this equation, we should have that equation. OK? And so we want these two equation to be equivalent.

So in order for them to be equivalent, you see the last term is the same. Because the m is just a number that commutes with D. And then in order for these two equations to be the same, we want D to commute-- it means that D should commute with gamma 0 but anti-commute with gamma i. OK? Because when you pass through here to bring it to that side of gamma i, you should change the sign. OK?

So now you can immediately write down such a matrix. Then D has to be-- it commutes with gamma 0, anti-commutes with gamma i. So it has to be proportional to gamma 0. OK? And then you can put the phase here. And you can put the phase here.

And so this is how it transforms. So now you can further constrain eta p. Now you can further constrain eta p. But here, the story is a little bit subtle. We are not going through there.

So you said the fermion-- remember when we already discussed in your quantum mechanics, when you have a spin-1/2 particle, when you rotate 360 degree, we actually don't have to go back to itself. You can go back with a minus sign. OK? And so here-- and so yeah.

And so here, there's an intricate story you can discuss what are allowed eta p etc. And so we will not go into that. But I think you know the basic idea. OK? Good. Any questions? Yes?

AUDIENCE: So isn't this just be, if you want these equations to be equivalent, that could be satisfied with gamma 0 is real gamma i is pure imaginary because of this antilinear?

HONG LIU: Wait. What do you mean by-- no, no, no. These equations should be applied for any gamma 0, for any gamma i.

AUDIENCE: Right. [INAUDIBLE]

HONG LIU: No, you cannot. Yeah, yeah. Because in parity, you should work in any representation.

AUDIENCE: OK.

HONG LIU: OK? So this is the story for the parity. So completely similar idea for charge conjugation and the time reversal for fermions. And so I will do it a little bit faster. OK? So I will do it a little bit faster-- mostly, again, just outlining the same idea, but then just write down the results.

So for the charge conjugation-- now let's do the charge conjugation. Did I say-- yeah, sorry. Here, I should put a thing here with parity. Because we are doing parity here. So now let's do the charge conjugation.

So for the scalar case for the charge conservation, just take the phi star. Again, so for the Dirac field for psi prime, again, this should be proportional to phi star, OK, to phi star. But so a simple way to construct phi star is you take the psi-bar. OK? And then you take a-- transpose. OK? And so this is some linear combination of the phi star.

And also, from what we discussed before, you should allow a matrix here. OK? You should allow a matrix here. And the C, again, is a matrix in the spinor space. OK?

And now, again, we need-- so the statement of the charge conjugation is the statement that, given there exists C, there exists C such that given equation star that phi prime will satisfy the same equation. OK? psi prime satisfies the same equation. OK?

So yeah. So we just do it. OK? So we just start from here. So since the psi prime are related by taking the bar, and the complex conj-- and transpose-- so let's then first do the bar for this equation and then do the transpose. OK?

So when you do the bar operation from that equation for the star, then give the bar equation [INAUDIBLE] psi bar-- yeah, you can easily check yourself-- equal to 0. OK? Now you do a transpose.

And then you get gamma mu t partial mu plus m psi t. OK. So then the psi-bar t is just equal to C minus 1 psi prime x equal to 0. OK-- because from here, by definition. OK?

So now you want this equation to be equivalent to that equation. OK? So as we do before-- a similar idea from that. OK? You just act C minus prime here. OK?

So in the end, you conclude that the gamma UT-- so from here, the equivalence of these two equations, you conclude that gamma mu t should be related to gamma mu by this C matrix. OK, negative. So if you can find the C, satisfy it to this equation, and then they are equivalent. OK? OK?

So does such a matrix C exist? So we can say actually such a matrix C always exists for the following reason. Because you can check given gamma mu satisfy this Clifford algebra for gamma matrices and the minus gamma mu t also satisfy the same algebra. OK? So this will do that.

So that means the minus gamma mu t, OK, is also another allowed set of gamma matrices. OK? We got to satisfy the same algebra. And now we discussed before, mathematically, we can prove all representations of the matrix are equivalent. And that means there must exist some matrix c which relates to these two. OK? And c must exist. OK? So this tells you c must exist.

So now it's not so easy to write down the C explicitly in this case. OK? So here, we can write the expression for D regardless of the choice of gamma matrices. And here, it's actually not so easy to write the explicit form of C. OK? So of course, for specific choice of representation of gamma, you can write down C. OK? But it's enough for us to know that C exists.

OK. And then this is the transformation for the charge conjugation. OK? It's the-- it transforms this way with C satisfy this condition. OK? C satisfy this condition. Are there any questions on this? OK. Good.

And then we'll talk about time reversal. The idea is, again, similar. So now we use UT. OK? So now suppose the psi prime is equal to, again, some matrix and then psi tx. OK?

This is the direct analog of this transformation. OK? And they're the same as the psi prime x prime equal to DT psi x. And now x prime is equal to Tx. OK? You just flip the time direction. You can just flip the time direction.

And now, again, the statement of the time reversal-- again, the statement of the time reversal--

Now the statement of the time reversal is that, given star, then it's again that the gamma mu partial mu prime minus m-- then you equal to psi prime x prime equal to 0. OK? Again, this new frame equal to 0. OK.

And then the story will be similar. OK? The story will be similar what you-- and then, yeah, you just again try to match starting from that equation. And then you try to match those equations. OK? So as I said, we'll not go into detail. And in the end, you find that DT should satisfy this equation. OK?

So you find DT should satisfy this equation. Our gamma mu t-- again, gamma mu transpose is equal to DT transpose gamma mu DT minus 1 transpose. OK? So gamma mu t has to satisfy this equation. OK?

So now if you compare these two equations-- that and this equation-- they just differ by a minus sign. OK? They just differ by a minus sign. So we conclude that we can do this by taking DT transpose equal to C minus 1 gamma 5.

So the gamma 5 will generate this minus sign. But when you commute with gamma mu, and then that will account for the extra minus sign. OK? That will account for the extra minus. So if you know C, then you also know this time translation symmetry matrix. OK? Yes?

AUDIENCE: [INAUDIBLE]

HONG LIU: Yeah, yeah. Yeah, because of the-- only the operator itself, only U is anti-linear. Right? t just goes right into the specific matrix acting on the spinner space. And so that's just an ordinary matrix. Yeah. Yeah, DT is just ordinary numbers. They're made up from the gamma matrices. Yeah. Other questions? Yes?

AUDIENCE: So is parity a response to the reflections in space?

HONG LIU: Yeah.

AUDIENCE: And time reversal-- is that a reflection in time?

HONG LIU: Yeah.

AUDIENCE: So can you, like, boost into spacetime? If you make a Lorentz boost, could space reflections then--

HONG LIU: No. No, they cannot relate these two. They are independent, discrete transformations. They're not related to a boost. Yeah. They are not related by a boost. Yeah, if they're related by a boost, then we should only need to worry about one of them. Because the boost we already understood.

Yeah. Yeah. So they are independent discrete transformations. They are not related by any other continuous transformation. Other questions? Yes?

AUDIENCE: There is only one parity transformation [INAUDIBLE].

HONG LIU: Yeah, that's right. Yeah, yeah, yeah, yeah. Just in any dimension, you have just one. Yeah. Good. Good. OK. Yes?

AUDIENCE: Yeah, I don't know if this makes sense. But is it possible to have a situation where in one frame a quantity is conserved through parity or time reversal transform, but then in a Lorentz boosted frame it's different than what [INAUDIBLE] or if it's conserved in one, it's conserved in all?

HONG LIU: Yeah. Yeah. If it's conserved in one, it's conserved in all of them. Yeah, it's because the conservation equation is a covariant equation. Yeah, because the conservation equation is partial mu, J mu mu equal to 0.

So this equation have the same form in any frame. Yeah. Yeah, but the charge is different in different frame. Yeah, they're related by the transformation. Just in any frame, you can define a conserved charge. Yeah. Good? OK.

So now let's move on to the next topic. So now, finally, we'll talk about path integrals for fermions. And with path integrals, then we can do interactions. OK? Remember, when we know how to do path integrals for scalar, then we can easily do interactions. And now let's do the path integral for fermion. And then that will give us a very simple way to treat interacting theories. OK?

So let's just recall in the path integral we considered before-- so if you have ordinary quantum mechanics-- so you have x and the p commutator equal to i. And then the path integral, you just integrate over all configurations of xt. OK?

So that's what you do in quantum mechanics. So now xt is just an ordinary function. OK? You just integrate over all possible trajectories. And xt is just an ordinary function.

So now when we go to quantum field theory, then we have phi then commute with-- yeah, yeah, yeah. So let me just put a hat just to show, to emphasize that this is our operator here. And the phi-- we have tx. And then we have pi phi, the conjugate momentum. And again, this is equal to i something, OK-- the delta function. OK?

OK. This is i. And then you have delta function. And then when you do the path integral, you just integrate over all configurations of phi xt. And now this is just a classical field. Now, essentially, you view this just as an ordinary function. OK? You integrate just all possible values of it, OK, all possible functions of it, OK, all possible functions of phi.

So now we have to do fermions. We have to do psi. So psi is a little bit weird. Because as we discussed before, the psi-- we no longer have the commutator. You actually have an anti-commutator. OK? So what do you do? OK? Do you still just follow the same rule, or do you actually have to change the rule? OK?

So that actually was a question which-- so Feynman actually-- he came up with this idea of the path integral in 1948. And then, of course, generalization to scalar will be immediate. OK?

But then he wanted to generalize to fermions. But then for a long time, he couldn't do it. And if you just use this method to do for fermions-- and they just didn't work. OK? We will not go through that exercise. If you does this for fermion, it just does not work.

So for many years, he couldn't find the answer. But then around the 1962-- I think it's 1962 or 1965. Anyway, just around some early 1960s, a former Soviet physicist actually came up with a brilliant idea to solve this problem, OK-- and, like, 15 years later after the discovery of path integral.

And so the basic idea is that when you have a Dirac field-- so when you have a Dirac field, then you have anti-commutation relation. psi and pi psi is an anti-commutation relation. OK? It's an anti-commutator.

And then this guy called Berezin. Berezin then postulates that, when you do the path integral, what is-- in the path integral will be an analog of the classical version of this field, OK, the classical version of this operator. So then what would be the classical version of this anti-commutator?

And he had just written it just should be some quantities which anti-commute. OK? So when we say classical, we put it in quotes because we only do this in the path integral. And we're then, of course, corresponding to anti-commuting fields, objects. OK?

And then when you do the path integral, you just integrate over psi. But psi is an anti-commuting object. OK? It turns out that this just brilliantly solve the problem. OK? It just immediately works. It's very simple.

So in hindsight, it's extremely simple. But actually, Feynman couldn't figure it out himself. OK. [LIGHT CHUCKLE] Right.

Anyway, so now let's talk a little bit about this anti-commuting object. OK? So these are called the Grassmann variables or numbers. So this Grassmann object, they just anti-commutes. OK? So they even anti-commute with themself.

So if you anti-commute with yourself, then the only thing you can have is a theta square that must be 0. OK? If you have anti-commuting object theta, and you can anti-commute yourself, it will be equal to 0. Because theta-squared should be equal to minus theta-squared. And then, of course, it can only be 0.

And so if you have two such objects, theta and eta-- and then equal to minus eta. And of course, eta-squared should also be 0. OK?

So this kind of property make this kind of object very simple. So now we can also talk about the functions of such objects. OK? We can also talk about the function of such objects. Say, if we have a function f of theta-- OK? So we define the function in terms of the power series. So suppose that we have an x-- say, if you write down a Gaussian function or whatever function, we define this function by the power series.

So power series means that to zeroth order, we have f0 corresponding to theta equal to 0, and then you have f1 theta. OK? But now if you look at f2 theta, theta-squared-- that would be 0.

So all higher-order terms can be 0. So you only have two terms. OK? So the only function of this variable will have two, terms, the constant term and the term proportional to theta. And then the functions of such variables are very simple. OK.

And if you have two variables, theta 1 and theta 2, then you just expand until you encounter-- you have f0 plus f1 theta 1 plus f2 theta 2 then plus f12 theta 1 theta 2. That's it. OK? That's it. Because for any other term we are involving either theta 1-squared or theta 2-squared, that will be 0.

So the differentiation will be extremely simple. So if you're taking d f theta d theta-- so you just-- because this is the constant. And so this is just equal to f1. OK.

But the one thing-- you do have to be careful when you have this multiple-variable function. And then it actually matters whether you take the derivative from the left or take the derivative from the right. OK? The direction you take the derivative becomes important. OK?

So for example, if we take the derivative-- yeah. Let me do it here. So if you, from the left, take the derivative f theta 1 on this f theta 1, theta 2, and then this term is 0. This term gives us f1.

This term does not depend on theta 1. And so this term is also 0 when we take the derivative. And here, we have theta 1. So we just get f12 theta 2. OK? So this is derivative from the left.

But now if I take the derivative from the right, theta 1, OK-- so this arrow means you take the derivative from the right. And again, the same thing for this one. It doesn't matter. There's only one theta 1. So you just get f1. Again, these two will be 0.

But now if we take the derivative from the right, then you have to move theta 1 to the right of theta 2 first because there's a theta 2 here. And then that gives you a minus sign. So it will give you f12 theta 2. So you differ by a minus sign. OK.

So from now on, without specifically mentioning, we always take derivative from the left. OK? So the convention is we always take the derivative of the left. Yes?

AUDIENCE: [INAUDIBLE]

HONG LIU: Sorry?

AUDIENCE: [INAUDIBLE] coefficient [INAUDIBLE] f0 [INAUDIBLE]

HONG LIU: What do you mean? Like, what type of object?

AUDIENCE: Like, [INAUDIBLE]

HONG LIU: Oh, yeah, yeah. They can be complex numbers. They can be real numbers. It's just-- f0, f1 is just some numbers. Yeah. For example-- yeah. So suppose f is a Gaussian function. OK?

And then you just write 1-- yeah, yeah. Say exponential theta. 1 plus theta. OK. Is exponential from just one-- just do Taylor expansion of theta. And the f0 and 1 just come from the Taylor expansion of whatever function you're looking at here. OK.

Yeah, similarly-- the same thing. 1 plus theta. So you have a function like that. So it's just 1 minus theta. Yeah, it's very simple. OK? So the functions of these variables are very simple. OK. Good.

So we talked about the functions. So we talked about the functions. Also, we talked about the-- yeah, the functions of them. We talked about derivatives. And then we need to talk about the integrals of them, OK-- so how to define integration.

So now let's talk about how to do integration. So how do you do integrals? Because we need to do the-- to be able to do path integrals, we need to integrate over such objects. OK? So to do integration, then we have to specify the rule for integration because it's no longer intuitive like a derivative. You can just take the derivative. OK?

So integration-- we have to specify the rule. And then the idea is you use the standard, the rule for integration, and then translate-- apply to this case. OK? So the rules we are going to require for the integration are the following.

So the first rule is that the integration should be linear. OK? So if I have, say, an integration of f theta-- I have two functions, f theta a plus g theta b. And a and b are some numbers. They can either be ordinary C-numbers. They also can be Grassmann numbers, which anti-commute. OK?

And then by linear-- but the integration operation is linear. This means that this is the same as the d theta f theta times a, where a just constant so you can just take it outside of the integral. d theta, g theta b. OK? So it should be linear, the operation.

And the second rule which we require for such integration is that the total derivative should be 0. OK? So the total derivative should be equal to 0. OK?

And turns out that these two conditions can be uniquely then it is enough to actually fix the other properties of the integral. OK? Yeah, it just can be used to fix the integral. Yes?

AUDIENCE: [INAUDIBLE] in not all derivatives [INAUDIBLE] when the boundary [INAUDIBLE]?

HONG LIU: Yeah. Here, we don't have a boundary. It's not required to be 0. Yeah, so theta-- yeah, we don't know how to specify the boundaries for theta. Yes?

AUDIENCE: How does [INAUDIBLE] multiply by it [INAUDIBLE]

HONG LIU: Yeah. You say this one?

AUDIENCE: No, [INAUDIBLE] symbol.

HONG LIU: Yeah. Sorry. Say it again?

AUDIENCE: Like, suppose [INAUDIBLE]

HONG LIU: Yeah, yeah, yeah, yeah, yeah. Indeed, it does not determine-- yeah, it only determines up to a constant. Yeah. Yeah, but then you can fix that constant. Yeah, I'm going to talk about it immediately. Yeah. Good?

So we have these two conditions. So let's see what these two conditions tell us. OK? So from this condition two, we conclude that d theta f2 is equal to 0 because of the-- the derivative of f theta-- or sorry, just f1 equal to 0. OK? We write-- yeah, just f1 equal to 0.

And since f1 is a C-number, we can take this f1 outside of the integral. So that means this is equal to 0. So that means that the theta 1 is equal to 0. OK? So just d theta any constant, it should be 0. OK. Just any constant, it should be 0. OK?

So this is one condition-- so one thing we deduce. OK? So now let's call this property 3 -- equation three this one follows from the one and two.

So now remember from the property of f theta equal to f0 plus f1 theta-- so that means d theta f theta is equal to d theta f0 plus f1 theta. OK? So that means-- so if we use this property one, you can take this constant outside it. OK? And this term is 0. So you just get the second term. You just get f1, d theta and the theta.

So now if we just fix the value of this object, then we fix the full integral. OK? And so you just fix this object to be 1. So we just fix-- define it to be 1. OK? Define up to a constant. We just define it be 1.

OK. So that's it. So that is fully specified to the rule. And then we conclude that d theta f theta is equal to f1.

So now you notice-- yeah, so this is the rule you should remember. So now you notice that this is the same as you take the derivative of theta. OK? So this story is a little bit funny. OK? So the integration of a function is the same as the derivative of the function. OK.

OK. That's it. So if the calculus for ordinary variables are such simple of-- [CHUCKLES]-- when we learn calculus, it will be much, much easier. But yeah. Yeah. So these Grassmann variables are very simple. OK. Any questions on this? Yes?

AUDIENCE: So what's the point in defining integration in this way if it ends up just being the same as the derivative one? Is it useful to have both integration and derivatives in [INAUDIBLE] if they did the same thing?

HONG LIU: Yeah. So this is for the single variable, right? When you go to the multivariable, then things are a little bit more intricate. Yeah. Yeah, but this rule will apply. This integration rule is general. Yes?

AUDIENCE: Why dont you define the integral as the opposite of the-- like, the reverse the derivative? Because now, for example, integrating and taking the derivative [INAUDIBLE]

HONG LIU: That's right. Yeah, just you don't know how to do that inverse operation anymore.

AUDIENCE: OK.

HONG LIU: Yeah. Just those kind of objects are sufficiently weird. Yeah, it's not easy. Yeah. So the basic idea, I think-- so you say, why Feynman didn't come up with this? Because if you just say, oh, I have to do anti-commuting, that's not enough. You have to invent the whole calculus for this kind of anti-commuting object.

And now the question is that, how do you invent such a calculus so that it's as natural as possible compared to the standard way? Yeah, yeah. And so I think this is a set of rule which seems to work very well. OK.

So yeah. Yeah. So when we say that Feynman missed it, it's not that he missed something that's just a trivial idea. He actually missed the whole calculus. OK.

So now let's just make a couple of remarks. So suppose eta is another Grassmann variable. OK? It means that it's also anti-commute. So now if we put eta before f theta-- OK?

And now, because eta have nothing to do with theta, you should be able to take it outside of the integral. But different variables anti-commute. So when you take eta outside the thing, the rule is that you can just take it out. It commutes through this d theta. You get a minus sign. So you just get the eta d theta f theta. OK? So this becomes just f1. OK.

So you can show that this rule, which I'm saying here, is compatible with also this rule. OK? It's also compatible with this rule because you can also commute the eta with f inside of the integral. And then you can use this rule. And you can show these two rules are compatible. OK? So this is the first remark.

The second remark is that the-- it's hard to do integration by parts. So now let's-- consider integral like this. a would be some ordinary-- a is just some number. OK?

So you can just expand this trivially. So this just gives-- you just find this f1 times a. OK? Just f1 times a. OK.

And of course, f1 times a is also the same as a times d theta prime f theta prime. If theta prime is a theta because it's a dummy variable. It doesn't matter. OK?

So this tells you-- so then when comparing these two-- OK. So this is like a change of variable. OK? Yes?

AUDIENCE: So if eta has to obey anti-commutation with d theta, why doesn't theta have to obey anti-commutation with [INAUDIBLE]?

HONG LIU: Yeah. Theta does have two. Yeah.

AUDIENCE: But then why [INAUDIBLE]?

HONG LIU: OK. But d theta and theta are two different objects, right? They anti-commute against each other. But this theta means the infinitesimal variation of theta, of course, even though we cannot quantify what this infinitesimal variation means. But these two are different object. Yeah. So their product is not-- yeah. OK.

Good. So this tells you-- so normally when we do a change of variable-- so this means that when you do the change of variable, theta prime is equal to a theta. And d theta prime is actually equal to 1 over a d theta, OK, if you compare these two. So now this is a very-- so now, again, this is opposite to the standard story. OK? So d theta prime is actually equal to 1 over a d theta. OK.

OK. So let's conclude here. Yeah. So our next lecture is now next Wednesday because Monday is a holiday. Yeah, it's going to be-- unfortunately, I hope you still remember this when we get to next Wednesday.