[SQUEAKING]

[RUSTLING]

[CLICKING]

HONG LIU: So in the last lecture, we have concluded the discussion of fermions. And now, we go to the last missing piece before we can talk about the QED. It's how to quantize the Maxwell field, OK? How to get photon, OK?

And so today, we start. OK? So this is a short chapter. I think we should be able to finish it just this week in two lectures. So first, let me just remind you of some aspects of the classical Maxwell theory. And then we will talk about this quantization, OK?

So the Lagrangian, say, for the classical Maxwell theory can be written as-- Lagrangian density can be written as F mu nu. So F mu nu is just the standard-- partial mu A nu equal to partial nu A mu. And the J mu is just the electromagnetic current.

OK. J mu just electromagnetic current. OK. And then so from this Lagrangian density, then you can derive the equation motion, which is just the Maxwell equation, OK? So the Maxwell equation is given by partial mu F mu nu equal to minus J mu, OK?

And so this is the familiar equation which we see from classical electrodynamics, OK? And so if you act partial mu on both sides-- so now, let's consider act partial mu on both sides. Since F mu mu is, by definition, antisymmetric, so this is automatically 0.

And so this tells you that by consistency, the current has to be conserved, OK? The current which is appearing on the right-hand side of the Maxwell equation has to be conserved. Otherwise, you don't have a consistent equations, OK?

And so another important feature of the Maxwell equation is so-called the gauge symmetry. So the Maxwell action-- both the action and the equation motion are invariant under-- so the L, yeah the action. So you integrate this over full spacetime.

So it's invariant under the following transformation-- that A mu goes to A mu prime equal to A mu plus partial mu lambda x. OK? And here the lambda x is an arbitrary function of spacetime coordinates, OK? It's an arbitrary function.

So the fact that the Maxwell theory-- the action is invariant under this transformation. So you can easily check yourself. Under this transformation, F mu mu does not change, OK? So the transformation between this and this-- they just cancel each other.

And so F mu nu does not change. And then you see the equation of motion, of course, is invariant under this transformation because F mu nu does not change. And J mu does not transform. And also, the action is invariant because the F mu nu does not change.

And naively, this term does change under the transformation because you have this partial mu lambda, but the additional term corresponding to partial mu lambda times J mu-- but then you can do integration by parts under the integration because J mu is conserved.

And so the action is invariant, OK? So the action is invariant. And so this symmetry is a little bit different from the symmetry we have discussed before because the symmetry depends on the function of arbitrary spacetime, of spacetime coordinates.

OK. So for this reason and the previous symmetry we have talked about, the transformation parameters are independent of the spacetime coordinates, OK? They're constants. And so that's why this is called the local symmetry. Sometimes, it's also called, just for historical reason, gauge symmetry.

OK. It's called local symmetry or gauge symmetry-- and yeah, in contrast to the global symmetry, which we encountered before, whose transformation parameters are spacetime-independent. So the gauge symmetry and the global symmetries-- even though they are all symmetries, they have very, very different physical interpretation, OK?

So global symmetry is a genuine symmetry. So the theory's translation variant means that what's happening here in Boston is the same happening in, say, Washington, DC, OK? Generally, things are invariant under that. But the local symmetry is very different, OK?

So the local symmetry means-- so the presence of gauge symmetry implies that the system has redundant degrees of freedom. OK?

So let me just give you a cartoon to illustrate this point. So the space of A mu, of course, is an infinite dimensional space. But let me just-- imagine this blackboard is the space of A mu, OK? This plane is the space of A mu.

Under this gauge transformation, it can be viewed as the following. Say for each A mu, imagine you make an arbitrary transformation-- this lambda x. And so this line-- so let's denote the space of lambda x. OK? So at each point A mu, you can make a transformation for different lambda corresponding to different points, OK?

And if you start with a different A mu, then you have a different-- yeah. So each of them is called-- so each of them is called the orbit of gauge transformations. OK? And they are parameterized by lambda, OK? And they're parameterized by lambda.

And the fact that the physics is invariant under this transformation means that the physics at this point for this choice of A is the same as for this choice of A, OK? And similarly, the physics on this joint-- just physics on the same orbit is the same, OK?

They describe the same physics. So in other words, only a cross-section-- so you can have many orbits, OK? So in each orbit, we can just choose a representative point. And then the physics only depends on a cross-section of all these different orbits, OK?

So physics only depends on a cross-section, OK? So a cross-section of this orbit, OK? Yes?

AUDIENCE: What's lambda?

HONG LIU: Yeah, yeah. Lambda. So different points corresponding to different lambda. So you view this, really, as an infinite dimensional space. And so each point corresponding to a different choice of lambda because lambda can be arbitrary function, OK?

And yeah. So yeah. Let me just parameterize by lambda x. OK. So the fact that we can choose this lambda arbitrarily-- so this means in this case, there's one scalar degree of freedom.

So lambda is just a scalar field. It can be viewed as a scalar field in spacetime coordinates. So this one scalar degree of freedom as parameterized by lambda is redundant. OK? Because any point on this orbit-- they carry exactly the same physics. They carry the exact same physics.

And so any questions on this? OK. So if you remember in classic E&M, when you solve the Maxwell equation, the physical object or the physical quantities-- they are all independent of the choice.

Yeah. So each choice of this section is called a gauge, OK? Choice of a gauge. And so the choice of a section is called the choice of a gauge. So when you solve the Maxwell equation, typically, you choose such a section. You choose a gauge.

And then you can just solve the Maxwell equation within that gauge so that you no longer have redundant degrees of freedom, OK? So often, this gauge is chosen so that you can solve the Maxwell equation, simplify the task of solving the Maxwell equation, OK?

So we often solve the Maxwell equation by fixing a gauge. OK. So there are two common examples which we use to solve the Maxwell equations. So one is called the Coulomb gauge. OK?

So in this gauge, you require the spatial components of A mu satisfy this equation, OK? So I will use the notation that A mu equal to phi-- so the zeroth component I call phi. And then the spatial component, the time component I call phi and the spatial component I just call A vector.

And also, I often write J mu as the rho, which is a charge density and then the electric current j, OK? So in this Coulomb gauge, then the Maxwell equation is simplified. OK? So you look at this equation. And you look at different components, OK?

And so we're not going into detail. Let me just write down the answer. When you impose this Coulomb gauge, you find the Maxwell equation becomes the following. So let me call this equation 1. Call this equation star. So in the Coulomb gauge, you find the equation star becomes-- so the time component becomes the so-called Poisson equation.

And then the spatial component becomes like this. OK? So you get these two equations. So let's just remind you a little bit of the physics encoded in these two equations, OK? So this is the equation in which we can use rho, the charge density distribution, to solve for the electric potential, for the scalar potential.

But notice that this equation does not involve time derivatives. OK? So this is not dynamical. So phi is not dynamical. It's determined by rho instantly. OK?

So this means that the configuration of phi at a given time is just determined by the configuration of rho exactly at that given time, OK? And there's no dynamics in the phi itself, OK? There's no time derivative in phi. There's no dynamic in the phi itself.

So this is first. The second thing-- is that let's see what this Coulomb gauge condition means. So if you look at this condition, when you go to momentum space-- so this Fourier transform goes to momentum space. And then that just become k dot A equal to 0, OK? So that just means that-- yes?

AUDIENCE: This is a technicality, but in the expression for A, shouldn't it be minus phi because it's A sub u, right?

HONG LIU: Which should be minus phi?

AUDIENCE: A sub u equals minus phi comma A. Because in the--

HONG LIU: Sorry. Which equation do you say is--

AUDIENCE: Oh, right under Coulomb gauge. In i. No, you're there.

HONG LIU: You mean here?

AUDIENCE: Yeah, above that. Yeah.

HONG LIU: No, this is my convention, right?

AUDIENCE: Oh.

HONG LIU: Yeah. This is my convention.

AUDIENCE: OK.

HONG LIU: Yeah. You can call it minus phi. You can call it positive phi. Yeah.

[CHUCKLING]

So in momentum space, you can just write it as this. So this just means that the component of A parallel to the momentum is 0, OK? A only has component which is perpendicular to the momentum and to the spatial momentum.

So we can separate, write A as the longitudinal plus A transverse, OK? The longitudinal is defined to be the component of A, which is proportional to k. So A i L-- when you go to the Fourier space, it will be proportional to k, OK?

And then the A i T will be transverse to k, means that A iT dot ki is equal to 0. Then the Coulomb gauge means-- this means that A is equal to A transpose, OK? Just A transpose-- just longitudinal part is 0, OK? The part proportional to the momentum is 0.

So now, we can already see what-- so I just combine these two features. Then how many dynamical degrees freedom does the Maxwell theory have? Yes?

AUDIENCE: Two.

HONG LIU: Why is it two?

AUDIENCE: Because phi is not dynamical, so you don't care about that.

HONG LIU: Yeah.

AUDIENCE: You have two transverse.

HONG LIU: That's right. The phi is not dynamical. We don't care. And then this Ai is 0. And so you only have the transverse component. The transverse component is perpendicular to the momentum direction. And you only have two independent components, OK?

So we find there are only two transverse dynamical degrees freedom. And indeed, this is the two polarizations. Classically, this is just corresponding two polarizations of electromagnetic wave, OK? So if you look at the EM wave, you only have two independent polarizations.

You only have two independent polarizations. So this is the story for the Coulomb gauge. So Coulomb gauge is very convenient and has been widely used-- in particular, in the nonrelativistic situations when you don't involve very fast velocity. But the Coulomb gauge, of course, is not perfect.

So there are some drawbacks of the Coulomb gauge. So first, there is no manifest Lorentz symmetry. So the Maxwell equation is manifestly Lorentz covariant, but the Coulomb gauge is not because this condition certainly is not covariant, OK?

It is not covariant. It's not the same in different frames, OK? And the one consequence of this loss of the manifest Lorentz covariance is this phi equation is determined instantaneously. So we know that when you have the relativistic theory, then everything should propagate smaller than the speed of light, OK?

And so you cannot have such instant action at a distance. So this implies the instant action at a distance, OK? And this is an artifact of the Coulomb gauge. This is an artifact of the Coulomb gauge. And the underlying physics is certainly not, OK?

So for example, one consequence of this is that the causality is not manifest. OK? Because you have instant action at a distance. But remember that phi itself is not physically observable. So we cannot observe phi physically.

So it's OK for phi to have an instant action. But of course, the electric field, the magnetic field-- they are causal. Yes?

AUDIENCE: So if we impose a Lorentz invariant condition on the gauge, do we not get any [INAUDIBLE]?

HONG LIU: Yeah, we are not. Yeah, yeah. We'll be manifest. Yeah. This also will not break the causality. Just the causality is not manifest because this phi is not physical, directly observable. Yeah. OK.

So then that motivates. So that's why the Coulomb gauge are mostly suitable in the case involving low velocities in the nonrelativistic situation. Yeah. So the second-- but in a more relativistic situation, people often consider so-called Lorentz gauge.

So the Lorentz gauge is defined to be partial mu A nu equal to 0, OK? So the partial mu x-- now, this equation is covariant because index are contracted. So this is a covariant equation, OK? So this is a covariant equation.

And so the Lorentz covariance is manifest. Then the Maxwell equation-- so under this gauge, then the equation star now becomes very simple. OK? It just becomes that. OK? So you see, this equation is indeed covariant, OK?

So the derivative contracted. And then you have J mu, OK? So now, this is something we are familiar-- this is like you have four independent scalar fields-- massless scalar fields-- but each of them has a source, OK? So that's the situation which you actually see in your pset, OK?

In Pset, you look at the scalar field with the source. And so this is like four independent scalar fields. Each is sourced by J mu. But of course, they are not independent because we have this gauge condition, OK? We have this gauge condition.

And also, this gauge condition, of course, is compatible with this equation because if you act partial mu on both sides, and then because of this gauge condition, the left side is 0 and the right-hand side is also 0 from the current conservation.

So now, let me also make some remarks on the Lorentz gauge. The Lorentz gauge is very convenient because it gives you seemingly decoupled equations between different A mu, OK? Between different A mu.

So the Lorentz gauge also suffers a little bit of a problem, OK? Inconvenience. Because the Lorentz gauge does not fix the gauge completely, OK? So after-- Lorentz gauge does not fix the gauge freedom completely.

So there's still some residual gauge symmetry left, OK? So consider A mu. Goes to A mu plus partial mu phi, OK? And if phi satisfies partial mu, partial nu equal to 0-- if phi satisfies this equation, then this preserves the gauge-- preserves the condition. Preserves this condition if phi satisfies the equation, OK?

So this tells you that the gauge freedom is not completely fixed, OK? So now, if you look at this freedom, so here it's like you can shift -- lambda is an arbitrary scalar field. It's an arbitrary function.

So you can view it as an arbitrary scalar field, OK? With no constraints, OK? With no constraint. We can say it's an arbitrary off-shell scalar field, OK? But now, after you fix the Lorentz gauge, you find that there's still some remaining gauge freedom-- and the remaining gauge freedom corresponding to an on-shell massive scalar field, OK?

Still, if phi satisfies this-- because this equation is like an equation of motion for a massless scalar field. And so after you fix the Lorentz gauge, there's still a freedom of the on-shell massless scalar field left, OK? So this is the first remark.

So the second remark is related to-- later, we are going to-- how we treat, say, this Lorentz gauge to quantize it, OK? So this is an alternative way. So you can just fix the Lorentz gauge, say, in your equation of motion.

Just in your equation of motion, anywhere, you see this thing. You set it to be 0, OK? But we can also actually impose the Lorentz gauge at the action level, OK? So we can fix the Lorentz gauge as the action level as follows.

So we consider a new action-- say, a new Lagrangian related to the previous one by a shift like this, OK? And the lambda here-- yeah. Let me not use lambda. So that should be not-- yeah, just xi, OK? So xi here is arbitrary constant.

OK? Xi is arbitrary constant. So now, let's look at this new theory we just obtained from the original theory by adding this term, OK? So now, let's look at this equation of motion. So if you look at this equation of motion, then you find it's given by the following form.

Partial square A mu. So partial square just means the partial nu partial nu, OK? And then you have lambda minus 1 partial mu-- partial nu A mu equal J mu, OK? You find the equation like this. OK. Yeah. Sorry. Sorry.

So you see an equation like this. So you get this new term because of in the Lagrangian, OK? So now, let's act partial mu on both sides. Since the current is conserved, and then when you act partial mu here, you get partial mu A mu.

And then yeah, you just combine together. You find you get partial square partial mu A mu equal to 0. So if you act the partial mu on both sides, you find that this equal to 0. So now, we can impose partial mu A mu equal to 0, OK?

We can enforce with star star by imposing boundary conditions so that this equation only has trivial solutions, which means that this partial mu A mu is equal to 0, OK? Sorry. I'm running out of space here, OK?

So you have this equation. And now, you can enforce the Lorentz gauge by putting the boundary conditions so that this equation only has 0 solution, OK? And then equivalently, then you have imposed the Lorentz gauge. But the nice thing of this approach is that now, you can treat it as action-level.

And the advantage, we will see later, OK? Because in quantum theory, the action is very important-- in particular, when you do the path integral. In particular, when you do the path integral. Any questions on this? Yes?

AUDIENCE: Could you go into a little bit more detail about what the specific [INAUDIBLE]?

HONG LIU: Yeah. We will do it later in your Pset.

[LAUGHTER]

Right. Yeah. So we will touch on this question. We will touch on this question when we quantize it. This feature will become very important. And then you will have an opportunity to work out in detail in your Pset. Other questions? Yes?

AUDIENCE: Is there a subtlety as to why you-- does it matter when you fix your gauge at the action level or later on with your equation of motion?

HONG LIU: Yeah. It's just a little bit easier. Yeah. Later, when we do the path integral, you will see it's easier. Yeah. Classically, there is no difference. But quantum mechanically, we always want to do is at the action level. Yeah.

AUDIENCE: [INAUDIBLE]

HONG LIU: Sorry?

AUDIENCE: [INAUDIBLE]

HONG LIU: Yeah. Yeah, we have an extra constant, but the physics should not depend on this constant. Other questions? OK, good. So let me just point out-- when xi is equal to 1, the story is particularly simple.

So for general xi, you get this more complicated equation, but for xi equal to 1, you notice that here this term is just 0. And then even with this new Lagrangian, you get exactly the same equation of motion as if you have imposed the Lorentz gauge in the equation of motion, OK? So xi equal to 1 is particularly simple.

So xi equal to 1 is particularly simple for a reason. So if you put the xi equal to 1 here, then you will find that this term actually cancel with some of the terms in this L exactly. So actually, when xi is equal to 1, you find that the Lagrangian becomes the following.

It becomes 1/2 partial mu A nu partial mu A nu minus J mu A nu, OK? It just becomes very simple. Again, just the action now becomes like you have just a bunch of free scalar fields, OK? Massless scalar fields. OK? Good. Any questions? Yes?

AUDIENCE: For the [INAUDIBLE]?

HONG LIU: Oh, sorry. Yeah. Yes, so this mu should be downstairs. OK. So just even at the action level, it looks like we have four decoupled massless scalar fields. OK, good. So now, we are ready. So this is a quick review of the classical story.

So now, let's discuss how to quantize it. OK. And so again, we will first do the standard canonical quantization, OK? Write down the most general solutions to the operator equations.

And then we will discuss using the path integral, OK? Using the path integral-- the path integral is convenient for treating interactions. And the canonical approach is convenient for understanding what's the physical degrees freedom, to understand what's going on physically, OK?

So for simplicity, let's just forget about J mu, OK? So let's just look at the Maxwell theory itself. So J mu does not do anything in terms of quantization of the Maxwell theory, OK? So we don't need to worry about it when we quantize this theory.

So this theory is quadratic in A mu. So this is what we call a free theory, OK? Because it's quadratic in A mu. There's no interaction, OK? So in the absence of source, in the absence of the charge density or current density, the Maxwell theory has no self-interaction, OK? Has no self-interaction.

So the photon is free in the absence of sources. So this is a free theory. But the quantization of this theory is actually very subtle due to the following reasons, OK? So subtleties in quantization.

First, is that because of the gauge symmetry, there's a redundant degree of freedom. OK? They are redundant degrees freedom, which are unphysical, OK? Which are unphysical. So we want to only quantize-- should quantize-- only physical degrees freedom.

OK? So this is much more subtle at the quantum level than the classical level. So at the classical level, we just fix the gauge, OK? Then that's it. But at the quantum level, things fluctuate, OK?

So you have to make sure the fluctuations you look at are corresponding to physical fluctuations, not corresponding to fluctuations of those unphysical degrees freedom, OK? And so that makes the quantum theory more subtle compared with classical, OK?

And the second subtlety is that F mu nu is antisymmetric. So the second one is more technical. So this one is very conceptual. It's antisymmetric. So that means there's no partial 0A0 term, OK? Because this is symmetric between the two indices.

And these are antisymmetric between two indices. There's no A0 partial 0 A0. Then that means there's no time derivative term for the A0 component in your action, OK? So that means L does not contain time derivatives of A0.

So this means that if we look at the momentum conjugate to A0-- we should look at the Lagrangian density divided by time derivative is 0-- this is just identically 0. So again, this is corresponding to a constrained equation.

This does not express a canonical momentum in terms of the A0 dot, OK? Or the time derivative of anything. So this is a constrained equation. And again, if you want to treat properly, we have to use constrained quantization, OK? We have to use constrained quantization.

And so this leads to subtleties, OK? Remember-- we said since the constrained allowing-- we already discussed for the Dirac equation. And there we avoided going into such constrained quantization by using a trick.

And here we will do similarly, OK? We will not directly deal with constrained quantization, but we use some trick to go around this issue. OK? Good.

And so let me just write down by passing the canonical momentum due to Ai, then this is Ai dot, OK? Dot is the time derivative. And then so this is corresponding to minus F0i. And then this gives you just Ei, OK? So the canonical momentum corresponding to Ai is just electric field.

OK. So it's just electric field. OK? So with this warning, then let's proceed to quantize the theory, OK? So as we said, we want to only quantize on a cross-section, OK? We don't want to quantize those unphysical degrees freedom, OK? Those unphysical degrees freedom.

And so we have to fix the gauge, OK? So we have to fix the gauge. OK? So first, let's see how to do it in the Coulomb gauge.

So Coulomb gauge is conceptually simple, OK? So in this case, so in the absence of source, let's look at the two equations. And then again, we follow the same strategy as before. We follow the same strategy as before.

We first find the most general classical solution. And then we turn that into the quantum operator solution, OK? The same strategy. So in the Coulomb gauge, then we have two equations. And then we just have phi squared equal to 0, and now partial mu A mu, partial mu, partial mu Ai now equal to partial phi, OK?

We have these two. So they're two equations with J equal to 0-- and then become the two equations, OK? And also, we have the gauge condition, which means that Ai must be transverse, OK? OK? So here are the equations we have to satisfy. OK. Here are the equations we have to satisfy.

So here in this gauge, we don't have this problem, which is a constrained quantization because phi is not a dynamic variable, OK? So we don't have to worry about quantizing it, OK? And in fact, if you look at this equation-- so this is so-called elliptic equation.

And then you can impose the boundary condition. So if you require phi to go to 0 at infinity-- spatial infinity-- and then this equation only has identical 0 solution. So the phi-- you can just set it to be 0. So phi is just 0, OK? And so now, since the phi is just 0, we don't have to worry about this canonical-- yeah. We just don't have to quantize phi.

And now, the equation for Ai becomes very simple. You just have partial mu, partial mu Ai equal to 0. And now, this just becomes massless scalar field equation. OK? This is just like you have a massless scalar field equation. OK?

So now, we can just proceed with the quantization. OK? But remember-- Ai is only transverse, OK? Ai has to be transverse. OK. So now, we impose this as an operator equation. So all this should be interpreted as an operator equation when we quantize it.

And so that means that Ai as an operator can only have a transverse component. So now, we can just quantize the theory. OK? So the solution to this, of course, we already know, OK? So we can just now proceed with the quantization.

So the first step is that we have to write down the canonical commutation relation. So this means that Ai with pi i, pi j, should be i delta ij, OK? We have to impose this, OK?

So this equation-- in the current context, if I write it carefully, it's the Ai t xi t, x, and then Ej t, x prime. So this should be equal to i delta ij delta x minus x prime. OK? Yes?

AUDIENCE: So for me to get to, to pick our gauge to start quantizing the theory, is there a reason why we are using the Coulomb gauge here? We discussed it violates causality--

HONG LIU: No, no, no. It does not violate causality, right? Just causality is not manifest. It does not violate causality.

AUDIENCE: What does that mean, not manifest?

HONG LIU: It means that the-- so manifest is at every step, you see the causality is preserved. And not manifest is that you have to check it to make sure causality is preserved. Yeah. So I say this equation is covariant.

I don't have to check it. I know this is covariant. I look at the equation. It's covariant, OK? But then if I look at this theory, I don't know that theory is covariant. But this theory is actually covariant. So the way to check if it's covariant is you check its observable quantities.

Yeah. But phi is not observable quantities. So that's why we say it's not manifestly causal. Yeah. So Coulomb gauge is a convenient gauge to use, even quantum mechanically. Yes?

AUDIENCE: So the canonical momentum here is not related to the field momentum E cross B, not related to-- we have the momentum carried by the field E cross B?

HONG LIU: No, no, no, no. No, it has nothing to do with that. No, no. This is the canonical momentum conjugated to the field. And that momentum you're talking about is the momentum carried by the electromagnetic field. It's the spacetime momentum carried by the electromagnetic field.

So that's the analog of the Noether charge. That momentum is the analog of the Noether charge. Yeah. For the scalar case, yeah. Yeah. Remember-- in the scalar case, there are also two momentums. Yeah. Other questions? OK, good.

So this is wrong, OK? So if we just naively write down this canonical commutation relation, this is wrong because this equation is incompatible with the Lorentz gauge condition, OK? So imagine you just add the derivative on this-- to the derivative on x.

And then since this is operator equation-- and then yeah. Yeah. Yeah. So let me call this equation 1. So if act partial i on 1, then the left-hand side, I have A t, x and Ej t, x prime. On the right-hand side, then I have the delta function.

Let me just-- partial i delta. OK? And the right-hand side is non-0. It's just a derivative of the delta function. But the left-hand side, according to this, should be 0, OK? So we have a contradiction. OK?

So this equation cannot be right. And the reason is simple. It's because this equation also includes the longitudinal degrees freedom. But we said-- Ai can only be transverse. But this equation actually is three components.

It also included the longitudinal. So we should not include that, OK? So what we should do-- we should only look at the Ai T, OK? So in fact, we should look at Ai T t, x and its conjugate momentum, which is called pi j T t, x prime.

So this should be i delta ij delta 3 x minus x prime, OK? So we should look at only the transverse component, OK? And the pi i-- the conjugate momentum for the transpose-- Ai-- if you start from here, so you convince yourself from here you can find it from yourself, OK?

This is just equal to partial 0 Ai T, OK? Just partial 0 Ai T, just the time derivative. So remember-- here Ei is equal to partial 0 Ai minus partial iA 0. And this part is 0 because phi is equal to 0. And then here if you take Ai to be transpose and then pi transpose to-- yeah, equal do that, OK?

But this equation is still not correct because the left-hand side is transverse, but the right-hand side is not transverse, OK? So here we have to impose a transverse projector. So rather than write it as delta ij, what I should do is I write it as a transverse projector.

OK? So Pij is that when you act the Ai T-- so Pij is if you act this on Aj, it projects into the Ai T. OK? So this is the transverse projector.

OK? And then now, this is a consistent equation, OK? Now, this is a consistent equation. OK? So this transverse projector, you can easily write down in momentum space. But in coordinate space, it's a little bit awkward to write it down.

So formally, in coordinate space, this Pi jT can be written as following-- as delta ij minus partial i partial j divided by delta squared. OK? And so you can easily understand the meaning of this equation because when you act on the-- yeah, yeah, yeah.

Anyway, you can check yourself. This works, OK? And to understand what this-- you can check formally yourself. This works, OK? Give you to the transpose. You can understand this equation by going to momentum space. You're going to momentum-- this just becomes ki, this kj. This just becomes k squared, OK?

This is just ki times kj divided by k squared, OK? And so the coordinate space definition of this is just corresponding to the Fourier transform of the momentum space expression, OK? Just formally write it as this, OK? Just formally write like this. Are there any questions on this? Yes?

AUDIENCE: So we don't really understand the operation of 1 over del squared. We just say this is the coordinate space representation of the momentum space?

HONG LIU: Yeah. Yeah, yeah. We just use this to denote-- use notation to denote what we mean by the Fourier transform of the momentum space expression. Yeah. Yes?

AUDIENCE: Do you mean the inverse of the Laplacian when you write 1 over grad squared?

HONG LIU: Huh?

AUDIENCE: Do you mean the inverse of the Laplacian when you write something something something over grad squared?

HONG LIU: Yeah. Yeah, this is just formal notation to make-- once you can check-- say, if you have something like this partial i Aj partial-- yeah. So this is a longitudinal part of A. Yeah, yeah, yeah. So one thing you can check yourself is, say, if you act this on the partial j Ai, partial j Ai-- and then this will cancel with the downstairs. And then that gives you the transverse part.

Yeah. Anyway, just try to treat this as a notation, OK? So momentum space is very easy to understand. OK. So this is now our canonical commutation relation. And now, we only quantize the physical degrees freedom, OK?

We only quantize the transverse degrees freedom. And everything else-- we don't worry about it, OK? Everything else-- we don't worry about it. And the transverse part just satisfies the standard massless equation of motion, OK? So remember-- the transverse part just satisfies the standard equation of motion.

So now, we can just immediately write down the most general expansion of Ai T. We just write down the most general theory that satisfies this-- yeah. This is just like a massless equation. So we can just write what we do previously.

OK? And then we can just write down-- so previously, we just say ak i k x plus ak dagger minus i k x. And the omega k in this case would be just equal to k, OK? Be good for massless. But now, here we have two independent components, OK?

So we have two independent components. So essentially, we have two, actually, independent solutions, OK? So now, let me parameterize the two independent solutions by a polarization vector-- to write more generally by polarization vector like this.

r equal to 1 to 2 epsilon i r, OK? And then this is r. This is r. OK? So epsilon r-- 1 to 2 are the basis of transverse vectors, OK? Because Ai has to be transverse, OK?

So essentially, you can treat it as polarization associated with this ak, OK? So by definition, this should satisfy, say, ir ki equal to 0. So this should be transverse, OK? So we also define them to be orthogonal to each other.

So r and s here are just 1 2, OK? So this is orthogonality normal to each other. Orthogonal, OK? And so they also satisfy the condition. So if you have two independent vectors, then when you sum them together, then you should get the projection operator, OK?

So you should get the Pij T k, OK? So epsilon i-- they are k dependent. They depend on the spatial momentum because they have to be orthogonal to the momentum. OK? And this is just the momentum space version of this. This is the delta ij. ki kj squared, OK?

So this is orthogonal condition. This is a completeness condition. OK? It's a completeness condition. Any questions on this? So now, this is the expansion for Ai, OK?

So the only difference from the scalar case-- is that now, here you have two independent vectors. And now, we have allowed the general polarization associated with the-- yeah. Have parameterized the general solution using two independent polarization vectors, OK?

So now, you can now plug this back into here. Now, you can find the commutation relation between and A and A dagger. Then you find, as you would expect, that they satisfy the standard. So from the commutation relation, we find that the akr and ak prime s dagger is equal to delta rs.

OK? With the rest 0, OK? So r and s equal to 1 2. And the others are 0. OK. OK. So now, we can write down the Hilbert space. We obtain the Hilbert space again just from what we do.

So we specify a vacuum requiring this-- just ar k annihilate the vacuum for any k and r equal to 1 and 2. And then the single particle state is given by just ak r and the dagger acting on 0.

So we denoted this as k and epsilon r, OK? So this describes a single photon state. This polarization vector given by epsilon r, OK? So remember-- epsilon r is the transverse vector, OK? And you have two independent of them. Yes?

AUDIENCE: Should there be some normalization, like square root 2 [INAUDIBLE]?

HONG LIU: Yeah. I believe so. Yeah, yeah. Let me just put the proportionality. Right, yeah. Thank you. OK. So yeah. So that concludes the story for quantization in the Coulomb gauge.

So just to summarize, for the Coulomb gauge so the Coulomb gauge is conceptually pretty simple, OK? So the advantage of the Coulomb gauge-- we said we can just directly quantize physical degrees of freedom, OK?

Because the unphysical degrees freedom, phi, just automatically is equal to 0. And then the longitudinal part, we can just throw away by hand using the Coulomb gauge by promoting the Coulomb gauge condition as the operator equation. And then we can just solve it by hand.

And then we just have the transverse part. And then once you have the transverse part, you can just quantize it then as a free massless field, OK? But the drawback, of course, is always not manifest Lorentz invariant, OK? Not manifest Lorentz covariance.

OK? Any questions on this? Good. So now, let's look at the Lorentz gauge. So these two are the most commonly used gauges. And also, these are two representatives, OK? They're two representative gauges because by quantizing them, they're involving completely different procedures.

And by contrasting what's happening in the Lorentz gauge-- and what's happening in the Lorentz gauge actually will be very instructive, OK? So we will just consider these two cases. So now, let's move on to the Lorentz gauge.

So in the Lorentz gauge, we will use a somewhat different strategy because in the Lorentz gauge-- so first is that solving the gauge condition becomes more difficult, OK? And also, in the Lorentz gauge, there is a residual gauge freedom, OK?

So there are two things. So in the Lorentz gauge, instead, we will start with this action. So we will start with this action. We will start with this action, which we say classically can be used to fix the Lorentz gauge, OK?

And when psi is equal to 1, so this just becomes partial mu A nu and partial mu A nu, OK? So this is for xi to 1. So you say, this theory-- we know how to quantize. These are just four decoupled massless scalar fields, OK?

For psi equal to 1, it's particularly simple. We just have four-- yeah. Oh, by the way, I should mention-- once you have that form, and once you have this commutation relation, and once you have this relation, you can find any correlation functions of A, OK? Two point functions-- propagator, Feynman propagator, retarded propagator. You can find all of them explicitly.

So now, this theory seems to be very simple. This just decoupled four massive scalar fields, OK? So we can just straightforwardly write down its canonical momentum. So pi nu-- this is just equal to A nu star, A nu dot. OK?

And the Hamiltonian density-- just given by 1/2 A nu dot A nu dot plus-- just like you have 4 independent massive scalar fields with the equation of motion given by this partial square A nu equal to 0, OK?

So now, you can just completely take over, OK? We can just completely take over what we did for the massless scalar, OK? To write down the answer for the massless scalar-- but we're running out of time today. So next time, we will talk about-- so now, you can just treat it as four massless scalar.

But then we have a problem, OK? Because we just need the Coulomb gauge. We only have two physical degrees freedom. But here we have four, OK? So somehow, we have to get rid of two, OK? And then we will find the ways to get rid of the two, OK? OK, yeah. So yeah, let's stop here for today.